Special Lagrangian 3-folds and integrable systems

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1 Introduction

This is the sixth in a series of papers [14, 15, 16, 17, 18] constructing explicit examples of special Lagrangian submanifolds (SL m-folds) in $\mathbb{C}^m$. The principal motivation for the series is to study the singularities of SL m-folds, especially when $m = 3$. This paper also has a second objective, which is to connect SL m-folds with the theory of integrable systems, and to arouse interest in special Lagrangian geometry within the integrable systems community.

We begin in §2 with a brief introduction to *special Lagrangian submanifolds* in $\mathbb{C}^m$, which are a class of real m-dimensional minimal submanifolds in $\mathbb{C}^m$, defined using calibrated geometry. Section 3 then gives a rather longer introduction to *harmonic maps* $\psi : S \to \mathbb{CP}^{m-1}$, where S is a Riemann surface. Such maps form an *integrable system*, and have a complex and highly-developed theory involving the Toda lattice equations, loop groups, and classification using spectral curves.

Section 4 explains the connection of this with special Lagrangian geometry. Let N be a *special Lagrangian cone* in $\mathbb{C}^3$, and set $\Sigma = N \cap \mathcal{S}^5$. Then Σ is a *minimal Legendrian surface* in $\mathcal{S}^5$, and so the image of a *conformal harmonic map* $\phi : S \to \mathcal{S}^5$ from a Riemann surface S. The projection $\psi = \pi \circ \phi$ of ϕ from $\mathcal{S}^5$ to $\mathbb{CP}^2$ is also conformal and harmonic, with Lagrangian image.

Thus, ψ can be analyzed in the integrable systems framework of §3. As the image of ψ is Lagrangian there is a simplification, in which the SU(3) Toda lattice equation reduces to the *Tzitzéica equation*, and the spectral curve acquires an extra symmetry. We use the integrable systems theory to give *parameter counts* for the expected families of SL T^2-cones in $\mathbb{C}^3$.

In §5 we give an explicit construction of special Lagrangian cones N in $\mathbb{C}^3$, involving two commuting o.d.e.s, and reducing to constructions given in [14, 15] in special cases. Taking the intersection with $\mathcal{S}^5$, we obtain families of explicit conformal harmonic maps $\phi : \mathbb{R}^2 \to \mathcal{S}^5$ and $\psi : \mathbb{R}^2 \to \mathbb{CP}^2$. Under some circumstances we can solve the conditions for these maps to be *doubly-periodic* in $\mathbb{R}^2$, and so to push down to harmonic maps $T^2 \to \mathcal{S}^5$ and $T^2 \to \mathbb{CP}^2$.

Section 6 analyzes this family of harmonic maps $\psi : \mathbb{R}^2 \to \mathbb{CP}^2$ from the point of view of integrable systems. We find that for generic initial data ψ is *superconformal*, and explicitly determine its harmonic sequence, Toda and Tzitzéica solutions, algebra of polynomial Killing fields, and spectral curve. In

§7 we generalize the ideas of §5 to give a new construction of special Lagrangian 3-folds in $\mathbb{C}^3$, which involves *three* commuting o.d.e.s, and reduces to the construction of §5 in a special case.

We end with an open problem. The SL 3-folds of §7 look very similar to those of §5, and share many of the hallmarks of integrable systems – commuting o.d.e.s, elliptic functions, conserved quantities. The author wonders whether these examples can also be explained in terms of some higher-dimensional integrable system, and indeed whether the special Lagrangian equations themselves are in some sense integrable.

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2 Special Lagrangian submanifolds in $\mathbb{C}^m$

We begin by defining *calibrations* and *calibrated submanifolds*, following Harvey and Lawson [12].

Definition 2.1 Let (M, g) be a Riemannian manifold. An *oriented tangent k-plane* V on M is a vector subspace V of some tangent space T_xM to M with $\dim V = k$, equipped with an orientation. If V is an oriented tangent k-plane on M then $g|_V$ is a Euclidean metric on V, so combining $g|_V$ with the orientation on V gives a natural *volume form* vol_V on V, which is a k-form on V.

Now let φ be a closed k-form on M. We say that φ is a *calibration* on M if for every oriented k-plane V on M we have $\varphi|_V \leqslant \mathrm{vol}_V$. Here $\varphi|_V = \alpha \cdot \mathrm{vol}_V$ for some $\alpha \in \mathbb{R}$, and $\varphi|_V \leqslant \mathrm{vol}_V$ if $\alpha \leqslant 1$. Let N be an oriented submanifold of M with dimension k. Then each tangent space T_xN for $x \in N$ is an oriented tangent k-plane. We say that N is a *calibrated submanifold* if $\varphi|_{T_xN} = \mathrm{vol}_{T_xN}$ for all $x \in N$.

It is easy to show that calibrated submanifolds are automatically *minimal submanifolds* [12, Th. II.4.2]. Here is the definition of special Lagrangian submanifolds in $\mathbb{C}^m$, taken from [12, §III].

Definition 2.2 Let $\mathbb{C}^m$ have complex coordinates $(z_1, \ldots, z_m)$, and define a metric g, a real 2-form ω and a complex m-form Ω on $\mathbb{C}^m$ by

$$\begin{gathered} g = |\mathrm{d}z_1|^2 + \cdots + |\mathrm{d}z_m|^2, \quad \omega = \frac{i}{2}(\mathrm{d}z_1 \wedge \mathrm{d}\bar{z}_1 + \cdots + \mathrm{d}z_m \wedge \mathrm{d}\bar{z}_m), \\ \text{and} \quad \Omega = \mathrm{d}z_1 \wedge \cdots \wedge \mathrm{d}z_m. \end{gathered} \tag{1}$$

Then $\operatorname{Re}\Omega$ and $\operatorname{Im}\Omega$ are real m-forms on $\mathbb{C}^m$. Let L be an oriented real submanifold of $\mathbb{C}^m$ of real dimension m, and let $\theta \in [0, 2\pi)$. We say that L is a *special Lagrangian submanifold* of $\mathbb{C}^m$ if L is calibrated with respect to $\operatorname{Re}\Omega$, in the sense of Definition 2.1. We will often abbreviate 'special Lagrangian' by

'SL', and 'm-dimensional submanifold' by 'm-fold', so that we shall talk about SL m-folds in $\mathbb{C}^m$.

As in [14] there is also a more general definition of special Lagrangian submanifolds involving a *phase* $\mathrm{e}^{i\theta}$, but we will not use it in this paper. Harvey and Lawson [12, Cor. III.1.11] give the following alternative characterization of special Lagrangian submanifolds.

Proposition 2.3 *Let L be a real m-dimensional submanifold of $\mathbb{C}^m$. Then L admits an orientation making it into an SL submanifold of $\mathbb{C}^m$ if and only if $\omega|_L \equiv 0$ and $\operatorname{Im}\Omega|_L \equiv 0$.*

Note that an m-dimensional submanifold L in $\mathbb{C}^m$ is called *Lagrangian* if $\omega|_L \equiv 0$. Thus special Lagrangian submanifolds are Lagrangian submanifolds satisfying the extra condition that $\operatorname{Im}\Omega|_L \equiv 0$, which is how they get their name.

3 Harmonic maps and integrable systems

A map $\phi : S \to M$ of Riemannian manifolds is *harmonic* if it extremizes the energy functional $\int_S |\mathrm{d}\phi|^2 \mathrm{d}V$. When S is 2-dimensional, the energy is conformally invariant, so that we may take S to be a Riemann surface. In this case, if ϕ is conformal, then ϕ is harmonic if and only if $\phi(S)$ is *minimal* in M. Thus, harmonic maps are closely connected to minimal surfaces.

We shall describe a relationship, due to Bolton, Pedit and Woodward [2], between a special class of harmonic maps $\psi : S \to \mathbb{CP}^{m-1}$ called *superconformal* harmonic maps, and solutions of the *Toda lattice equations* for $\mathrm{SU}(m)$. Then we will explain how superconformal maps can be studied using *loop groups* and *loop algebras*.

This leads to the definition of *polynomial Killing fields* and a special class of superconformal maps called *finite type*, which include all maps from T^2. Finally we explain how to associate a Riemann surface called the *spectral curve* to each finite type superconformal map, and that finite type harmonic maps can be classified in terms of algebro-geometric data including the spectral curve.

This is a deep and complex subject, and we cannot do it justice in a few pages. A good general reference on the following material is Fordy and Wood [10], in particular, the articles by Bolton and Woodward [10, p. 59–82], McIntosh [10, p. 205–220] and Burstall and Pedit [10, p. 221–272].

3.1 The harmonic sequence and superconformal maps

Suppose S is a connected Riemann surface and $\psi : S \to \mathbb{CP}^{m-1}$ a harmonic map. Then the *harmonic sequence* (ψ_k) of ψ is a sequence of harmonic maps $\psi_k : S \to \mathbb{CP}^{m-1}$ with $\psi_0 = \psi$, defined in Bolton and Woodward [3, §1]. Each $\psi_k : S \to \mathbb{CP}^{m-1}$ defines a holomorphic line subbundle L_k of the trivial vector bundle $S \times \mathbb{C}^m$, where a section s of L_k is defined to be holomorphic if $\partial s/\partial\bar{z}$ is orthogonal to L_k.

The ψ_k and L_k are characterized by the following property. If U is an open subset of S and z a holomorphic coordinate on U, then any nonzero holomorphic section ϕ_0 of L_0 over U may be extended uniquely to a sequence of nonzero holomorphic sections ϕ_k of L_k over U satisfying

$$\begin{aligned} \langle \phi_k, \phi_{k+1} \rangle = 0, \quad \frac{\partial \phi_k}{\partial z} = \phi_{k+1} + \frac{\partial}{\partial z}\big(\log|\phi_k|^2\big)\phi_k \\ \text{and} \quad \frac{\partial \phi_k}{\partial \bar{z}} = -\frac{|\phi_k|^2}{|\phi_{k-1}|^2}\phi_{k-1} \quad \text{for all } k, \end{aligned} \tag{2}$$

where $\langle\,,\,\rangle$ is the standard Hermitian product on $\mathbb{C}^m$. (Actually one should allow the ϕ_k to be meromorphic, but we will ignore this point.) If (ϕ_k), (ϕ'_k) both satisfy (2) then $\phi'_k = f\phi_k$ for some holomorphic $f : U \to \mathbb{C}^*$ and all k, where $\mathbb{C}^* = \mathbb{C} \setminus \{0\}$. Thus $\psi_k = [\phi_k]$ is independent of the choice of ϕ_0.

Note that ψ_k may not be defined for all $k \in \mathbb{Z}$. For if $\psi_k : S \to \mathbb{CP}^{m-1}$ is holomorphic then $\frac{\partial \phi_k}{\partial z} = 0$, so that $\phi_{k+1} = 0$ and ψ_{k+1} is undefined, and the sequence terminates above at ψ_k. Similarly, if ψ_k is antiholomorphic then ψ_{k-1} is undefined, and the sequence terminates below at ψ_k.

If ψ_k exists for all $k \in \mathbb{Z}$ then ψ is called *non-isotropic*. Otherwise ψ is called *isotropic*. Isotropic maps $\psi : S \to \mathbb{CP}^{m-1}$ were studied by Eells and Wood [8]. They all arise by projection from certain holomorphic maps into a complex flag manifold, and so are fairly easy to understand and construct.

Harmonic sequences have strong orthogonality properties. Two points in $\mathbb{CP}^{m-1}$ are called *orthogonal* if the corresponding lines in $\mathbb{C}^m$ are orthogonal at all points, and two maps $\psi_j, \psi_k : S \to \mathbb{CP}^{m-1}$ are called orthogonal if $\psi_j(s)$ and $\psi_k(s)$ are orthogonal in $\mathbb{CP}^{m-1}$ for all $s \in S$.

Bolton and Woodward [3, Prop. 2.4] show that if some set of l consecutive terms in a harmonic sequence (ψ_k) are mutually orthogonal, then *every* set of l consecutive terms of (ψ_k) are mutually orthogonal. A harmonic map $\psi : S \to \mathbb{CP}^{m-1}$ and its harmonic sequence (ψ_k) are both called *l-orthogonal* if every set of l consecutive terms are mutually orthogonal.

Clearly, every harmonic sequence is 2-orthogonal. It is easy to show that $\psi = \psi_0$ is conformal if and only if ψ_1 and ψ_{-1} are orthogonal. Therefore, ψ is 3-orthogonal if and only if it is conformal, and then all the elements ψ_k of the harmonic sequence are also conformal.

The maximum number of mutually orthogonal elements of $\mathbb{CP}^{m-1}$ is m, and so a harmonic map $\psi : S \to \mathbb{CP}^{m-1}$ is at most m-orthogonal. A nonisotropic, m-orthogonal harmonic map $\psi : S \to \mathbb{CP}^{m-1}$ is called *superconformal*. The harmonic sequence of a superconformal map ψ is *periodic*, with period m, so that $\psi_{k+m} = \psi_k$ for all k.

A nonisotropic, conformal harmonic map $\psi : S \to \mathbb{CP}^2$ is superconformal, as ψ is 3-orthogonal because it is conformal, from above. Thus, every conformal harmonic map $\psi : S \to \mathbb{CP}^2$ is either isotropic or superconformal.

3.2 The Toda lattice equations

The *Toda lattice equations* for $\mathrm{SU}(m)$ may be written as follows. For all $k \in \mathbb{Z}$, let $\chi_k : \mathbb{C} \to (0,\infty)$ be differentiable functions satisfying

$$\chi_{k+m} = \chi_k \quad \text{for all } k \in \mathbb{Z}, \quad \chi_0\chi_1\cdots\chi_{m-1} \equiv 1, \quad \text{and} \tag{3}$$

$$\frac{\partial^2}{\partial z \partial \bar{z}}\big(\log \chi_k\big) = \chi_{k+1}\chi_k^{-1} - \chi_k\chi_{k-1}^{-1} \quad \text{for all } k \in \mathbb{Z}. \tag{4}$$

They are important integrable equations in mathematical physics, and large classes of solutions to them may be constructed using loop algebra methods.

We shall show how to construct a solution of (3)–(4) from a superconformal map $\psi : S \to \mathbb{CP}^{m-1}$. Use the notation of §3.1, and suppose ψ is superconformal. Define functions $\chi_k : U \to (0,\infty)$ by $\chi_k = |\phi_k|^2$. Using the fact that $\partial^2\phi_k/\partial z\partial\bar{z} = \partial^2\phi_k/\partial\bar{z}\partial z$, one can show using (2) that

$$\frac{\partial^2}{\partial z \partial \bar{z}} \log|\phi_k|^2 = \frac{|\phi_{k+1}|^2}{|\phi_k|^2} - \frac{|\phi_k|^2}{|\phi_{k-1}|^2}.$$

Thus the χ_k satisfy (4).

To make the χ_k satisfy (3) as well, we need to choose the coordinate z and lifts ϕ_k more carefully. As ψ is superconformal, the ψ_k are periodic with period m. We shall arrange for the lifts ϕ_k also to be periodic with period m. Then $\chi_{k+m} = \chi_k$ for all k, the first equation of (3). This can be done by a suitable choice of holomorphic coordinate z.

It is not difficult to show that the ϕ_k satisfy $\phi_{k+m} = \xi\phi_k$ for all k, where ξ is a nonzero holomorphic function on U. If we change to a new holomorphic coordinate z' on U, then ξ is replaced by

$$\xi' = \left(\frac{\partial z'}{\partial z}\right)^{-m}\xi. \tag{5}$$

Thus, by changing coordinates we can arrange that $\xi \equiv 1$, so that $\phi_{k+m} = \phi_k$ for all k, as we want. A holomorphic coordinate z on an open subset U of S with this property is called *special* [10, p. 65]. Such coordinates are unique up to addition of a constant, and multiplication by an m^{th} root of unity.

Suppose from now on that z is special, so that $\phi_{k+m} = \phi_k$ for all k. It remains to show that we can choose the ϕ_k such that the second equation of (3) holds. Regard the ϕ_k as complex column vectors, so that $(\phi_0\,\phi_1\cdots\phi_{m-1})$ is a complex $m\times m$ matrix. Then the determinant $\det(\phi_0\,\phi_1\cdots\phi_{m-1})$ is a nonzero holomorphic function on U.

From above, the ϕ_k are defined uniquely up to multiplication by some holomorphic function $f : U \to \mathbb{C}^*$. By multiplying the ϕ_k by a suitable f we can arrange that

$$\det(\phi_0\,\phi_1\cdots\phi_{m-1}) \equiv 1. \tag{6}$$

This fixes the ϕ_k uniquely up to multiplication by an m^{th} root of unity. As ψ is superconformal, $\phi_0, \ldots, \phi_{m-1}$ are complex orthogonal in $\mathbb{C}^m$. It follows that

$$\chi_0\chi_1\cdots\chi_{m-1} \equiv |\phi_0|^2|\phi_1|^2\cdots|\phi_{m-1}|^2 \equiv \left|\det(\phi_0\,\phi_1\cdots\phi_{m-1})\right|^2 \equiv 1,$$

so that the second equation of (3) holds. Thus equations (3)–(4) hold, and the χ_k satisfy the Toda lattice equations for $\mathrm{SU}(m)$.

When S is a torus T^2 and $\psi : S \to \mathbb{CP}^{m-1}$ a superconformal harmonic map, from [2, Cor. 2.7] and [10, p. 67-8] there exists a global special holomorphic coordinate z on the universal cover $\mathbb{C}$ of T^2, which then yields a solution (χ_k) of the Toda lattice equations (3)–(4) on the whole of $\mathbb{C}$. In particular, there are no 'higher order singularities', and the ϕ_k and ξ do not have zeros or poles.

3.3 Toda frames and the reconstruction of ψ

Above we saw how to construct a solution (χ_k) of the Toda lattice equations for $\mathrm{SU}(m)$ out of a superconformal harmonic map $\psi : S \to \mathbb{CP}^{m-1}$. We shall now explain how to go the other way, and reconstruct ψ from (χ_k). We continue to use the same notation.

Define $F : U \to \mathrm{GL}(m, \mathbb{C})$ by $F = (f_0\, f_1 \cdots f_{m-1})$, where $f_j = |\phi_j|^{-1}\phi_j$. Then as $\phi_0, \ldots, \phi_{m-1}$ are complex orthogonal and $\det(\phi_0\,\phi_1\cdots\phi_{m-1}) = 1$, F actually maps $U \to \mathrm{SU}(m)$. We call F a *Toda frame* for ψ on U, [2, p. 126]. Define α to be the matrix-valued 1-form $F^{-1}\mathrm{d}F$ on U. Then by (2) we find that α is given by

$$\begin{pmatrix} -\frac{i}{2}J\mathrm{d}\log\chi_0 & -\chi_1^{1/2}\chi_0^{-1/2}\mathrm{d}\bar z & & & \chi_0^{1/2}\chi_{m-1}^{-1/2}\mathrm{d}z \\ \chi_1^{1/2}\chi_0^{-1/2}\mathrm{d}z & -\frac{i}{2}J\mathrm{d}\log\chi_1 & \ddots & & \\ & \chi_2^{1/2}\chi_1^{-1/2}\mathrm{d}z & \ddots & -\chi_{m-2}^{1/2}\chi_{m-3}^{-1/2}\mathrm{d}\bar z & \\ & & \ddots & -\frac{i}{2}J\mathrm{d}\log\chi_{m-2} & -\chi_{m-1}^{1/2}\chi_{m-2}^{-1/2}\mathrm{d}\bar z \\ -\chi_0^{1/2}\chi_{m-1}^{-1/2}\mathrm{d}\bar z & & & \chi_{m-1}^{1/2}\chi_{m-2}^{-1/2}\mathrm{d}z & -\frac{i}{2}J\mathrm{d}\log\chi_{m-1} \end{pmatrix}, \tag{7}$$

where J is the complex structure on U.

Now α is a 1-form on U with values in $\mathfrak{su}(m)$, so we may regard it as a *connection 1-form* upon the trivial $\mathrm{SU}(m)$-bundle over U. The connection $\mathrm{d}+\alpha$ is automatically *flat*, as α is of the form $F^{-1}\mathrm{d}F$, so that α satisfies

$$\mathrm{d}\alpha + \tfrac{1}{2}[\alpha\wedge\alpha] = 0. \tag{8}$$

Furthermore, α depends only on the solution (χ_k) of the Toda equations.

Thus, to reconstruct ψ from (χ_k), we proceed as follows. Given (χ_k), we can write down the flat $\mathfrak{su}(m)$-connection $\mathrm{d}+\alpha$ on U. Then we retrieve the Toda frame $F : U \to \mathrm{SU}(m)$ by solving the equation $\mathrm{d}F = F\alpha$, which is in effect two commuting first-order linear o.d.e.s. If U is simply-connected there exists a solution F, which is unique up to multiplication $F \mapsto AF$ by some $A \in \mathrm{SU}(m)$.

Finally we define $\psi = [f_0]$, where f_0 is the first column of F. In this way, any solution (χ_k) of the Toda lattice equations on a simply-connected open set U in $\mathbb{C}$ generates a superconformal map $\psi : U \to \mathbb{CP}^{m-1}$, which is unique up to multiplication by $A \in \mathrm{SU}(m)$, that is, up to automorphisms of $\mathbb{CP}^{m-1}$.

3.4 Loop groups and loops of flat connections

A large part of the integrable systems literature on harmonic maps is formulated in terms of infinite-dimensional Lie groups known as *loop groups*. If G is a finite-dimensional Lie group, the *loop group* LG is the group of smooth maps $\mathcal{S}^1 \to G$, under pointwise multiplication and inverses, and the corresponding *loop algebra* $L\mathfrak{g}$ is the Lie algebra of smooth maps $\mathcal{S}^1 \to \mathfrak{g}$, where $\mathfrak{g}$ is the Lie algebra of G.

In the situation of §3.3, for each $\lambda \in \mathbb{C}$ with $|\lambda| = 1$, define an $\mathfrak{su}(m)$-valued 1-form α_λ on U to be

$$\begin{pmatrix} -\frac{i}{2}J\mathrm{d}\log\chi_0 & -\lambda\chi_1^{1/2}\chi_0^{-1/2}\mathrm{d}\bar{z} & & & \lambda^{-1}\chi_0^{1/2}\chi_{m-1}^{-1/2}\mathrm{d}z \\ \lambda^{-1}\chi_1^{1/2}\chi_0^{-1/2}\mathrm{d}z & -\frac{i}{2}J\mathrm{d}\log\chi_1 & \ddots & & \\ & \lambda^{-1}\chi_2^{1/2}\chi_1^{-1/2}\mathrm{d}z & \ddots & -\lambda\chi_{m-2}^{1/2}\chi_{m-3}^{-1/2}\mathrm{d}\bar{z} & \\ & & \ddots & -\frac{i}{2}J\mathrm{d}\log\chi_{m-2} & -\lambda\chi_{m-1}^{1/2}\chi_{m-2}^{-1/2}\mathrm{d}\bar{z} \\ -\lambda\chi_0^{1/2}\chi_{m-1}^{-1/2}\mathrm{d}\bar{z} & & & \lambda^{-1}\chi_{m-1}^{1/2}\chi_{m-2}^{-1/2}\mathrm{d}z & -\frac{i}{2}J\mathrm{d}\log\chi_{m-1} \end{pmatrix}. \tag{9}$$

When $\lambda = 1$ this coincides with the 1-form α of (7). Using the Toda lattice equations one can show that $\mathrm{d} + \alpha_\lambda$ is also a flat $\mathrm{SU}(m)$-connection, so that

$$\mathrm{d}\alpha_\lambda + \tfrac{1}{2}[\alpha_\lambda \wedge \alpha_\lambda] = 0. \tag{10}$$

Thus the family $\{\alpha_\lambda\}$ gives a *loop of flat connections*. We can interpret this in loop group terms as follows. We defined the α_λ as an $\mathcal{S}^1$ family of 1-forms on $U \subseteq \mathbb{C}$ with values in $\mathfrak{su}(m)$, but we can instead regard it as a single 1-form on U with values in the loop algebra $L\mathfrak{su}(m)$. So the α_λ give an $L\,\mathrm{SU}(m)$-connection on U, which turns out to be flat.

If U is simply-connected, there exists a smooth 1-parameter family of maps $F_\lambda : U \to \mathrm{SU}(m)$ with $F_\lambda^{-1}\mathrm{d}F_\lambda = \alpha_\lambda$, which are unique up to multiplication $F_\lambda \mapsto A_\lambda F_\lambda$ by elements $A_\lambda \in \mathrm{SU}(m)$. The family $\{F_\lambda\}$ is called an *extended Toda frame* for ψ. In loop group terms, we may interpret the F_λ as a map $U \to L\,\mathrm{SU}(m)$. It turns out that each F_λ is the Toda frame of a superconformal harmonic map $\psi_\lambda : U \to \mathbb{CP}^{m-1}$, where the special holomorphic coordinate on U is $\lambda^{-1}z$ rather than z.

Now if $\Phi : U \to L\,\mathrm{SU}(m)$ is any smooth map, then $\mathrm{d} + \Phi^{-1}\mathrm{d}\Phi$ is a flat $L\,\mathrm{SU}(m)$-connection on U, or equivalently a loop of flat $\mathrm{SU}(m)$-connections on U. This gives an enormous family of loops of flat $\mathrm{SU}(m)$-connections on U, most of which have nothing to do with harmonic maps into $\mathbb{CP}^{m-1}$. The important thing about the family $\{\alpha_\lambda\}$ is that it has two special algebraic properties.

The first property is that we may write α_λ in the form

$$\alpha_\lambda = (\alpha'_1\lambda + \alpha'_0)\mathrm{d}z + (\alpha''_{-1}\lambda^{-1} + \alpha''_0)\mathrm{d}\bar{z}, \tag{11}$$

where $\alpha'_1, \alpha'_0, \alpha''_{-1}$ and α''_0 map $U \to \mathfrak{su}(m)^{\mathbb{C}} = \mathfrak{sl}(m, \mathbb{C})$. This equation says two things. Firstly, as a Laurent series in λ we have $\alpha_\lambda = \alpha_1\lambda + \alpha_0 + \alpha_{-1}\lambda^{-1}$. Secondly, if we decompose α_λ into $(1,0)$ and $(0,1)$ parts as $\alpha_\lambda = \alpha'_\lambda \mathrm{d}z + \alpha''_\lambda \mathrm{d}\bar{z}$, then $\alpha'_\lambda = \alpha'_1\lambda + \alpha'_0$, so that α'_λ has no λ^{-1} component, and $\alpha''_\lambda = \alpha''_{-1}\lambda^{-1} + \alpha''_0$, so that α''_λ has no λ component.

The second property is this. Define $\zeta = \mathrm{e}^{2\pi i/m}$, so that $\zeta^m = 1$, and let Υ be the diagonal $m \times m$ matrix with entries $1, \zeta^{-1}, \zeta^{-2}, \ldots, \zeta^{1-m}$. Then

$$\alpha_{\zeta\lambda} = \Upsilon \alpha_\lambda \Upsilon^{-1} \quad \text{for all } \lambda \in \mathbb{C} \text{ with } |\lambda| = 1. \tag{12}$$

That is, α_λ is equivariant under $\mathbb{Z}_m$-actions on $\mathcal{S}^1$ and $\mathfrak{su}(m)$.

It follows from Bolton, Pedit and Woodward [2, §2] that an $\mathcal{S}^1$-family $\mathrm{d} + \alpha_\lambda$ of flat $\mathrm{SU}(m)$-connections on a simply-connected open subset $U \subseteq \mathbb{C}$ come from a solution of the Toda lattice equations, and hence from a superconformal map $\psi : U \to \mathbb{CP}^{m-1}$, if and only if the α_λ satisfy (11)–(12) and the additional condition that $\det(\alpha'_1)$ is nonzero except at isolated points in U.

3.5 Polynomial Killing fields

Polynomial Killing fields were introduced by Ferus et al. [9, §2] and used extensively by Burstall et al. [6], but in a somewhat different situation to us. Our treatment is based on McIntosh [24, App. A] and Bolton, Pedit and Woodward [2, §3].

We shall work with the Lie algebra $\mathfrak{u}(m)$ and its complexification $\mathfrak{gl}(m, \mathbb{C})$ rather than $\mathfrak{su}(m)$ and $\mathfrak{sl}(m, \mathbb{C})$. Let ζ and Υ be as in §3.4. For each $d \in \mathbb{N}$, let $\Lambda_d \mathfrak{gl}(m, \mathbb{C})$ be the vector space of maps $\eta : \mathbb{C}^* \to \mathfrak{gl}(m, \mathbb{C})$ of the form $\eta(\lambda) = \sum_{n=-d}^{d} \eta_n \lambda^n$, where $\eta_n \in \mathfrak{gl}(m, \mathbb{C})$, which satisfy

$$\eta(\zeta\lambda) = \Upsilon \eta(\lambda) \Upsilon^{-1} \quad \text{for all } \lambda \in \mathbb{C}^*. \tag{13}$$

That is, η is equivariant under the same $\mathbb{Z}_m$-actions as α_λ in (12).

Let $\Lambda_d \mathfrak{u}(m)$ be the real vector subspace of $\eta \in \Lambda_d \mathfrak{gl}(m, \mathbb{C})$ such that $\eta(\lambda)$ lies in $\mathfrak{u}(m)$ for all $\lambda \in \mathbb{C}$ with $|\lambda| = 1$. Then $\Lambda_d \mathfrak{gl}(m, \mathbb{C}) = \Lambda_d \mathfrak{u}(m) \otimes_{\mathbb{R}} \mathbb{C}$. Note that by restricting η to $\mathcal{S}^1$ in $\mathbb{C}^*$, we can regard $\Lambda_d \mathfrak{gl}(m, \mathbb{C})$ and $\Lambda_d \mathfrak{u}(m)$ as finite-dimensional vector subspaces of the loop algebras $L\mathfrak{gl}(m, \mathbb{C})$ and $L\mathfrak{u}(m)$.

We define a *polynomial Killing field* on U to be a map $\eta : U \to \Lambda_d \mathfrak{gl}(m, \mathbb{C})$ for some $d \in \mathbb{N}$ satisfying

$$\mathrm{d}\eta = [\eta, \alpha_\lambda]. \tag{14}$$

We call η *real* if it maps to $\Lambda_d \mathfrak{u}(m)$ in $\Lambda_d \mathfrak{gl}(m, \mathbb{C})$. We may write $\eta = \eta(\lambda, z)$ for $\eta \in \mathbb{C}^*$ and $z \in U$, and decompose η as

$$\eta(\lambda, z) = \sum_{n=-d}^{d} \eta_n(z) \lambda^n, \quad \text{where } \eta_n \text{ maps } U \to \mathfrak{gl}(n, \mathbb{C}). \tag{15}$$

Using the decompositions (11) and (15) of α_λ and η, it is easy to show that (14) is equivalent to the equations

$$\frac{\partial \eta_n}{\partial z} = [\eta_n, \alpha'_0] + [\eta_{n-1}, \alpha'_1] \quad \text{and} \tag{16}$$

$$\frac{\partial \eta_n}{\partial \bar{z}} = [\eta_n, \alpha''_0] + [\eta_{n+1}, \alpha''_{-1}] \quad \text{for all } n, \tag{17}$$

where we set $\eta_n \equiv 0$ if $|n| > d$.

Define $\mathcal{A}$ to be the vector space of polynomial Killing fields. It is easy to see that the polynomial Killing fields form a Lie algebra under the obvious Lie bracket. In our case, where ψ is superconformal, this Lie algebra is *abelian*, and the polynomial Killing fields form a *commutative algebra* under matrix multiplication [24, p. 240-1]. The reason why we work with $\mathfrak{gl}(m, \mathbb{C})$ rather than $\mathfrak{sl}(m, \mathbb{C})$ is because $\mathfrak{sl}(m, \mathbb{C})$ is not closed under matrix multiplication.

Following [2, p. 133], we say that ψ is of *finite type* if there exists a real polynomial Killing field $\eta : U \to \Lambda_d \mathfrak{u}(m)$ for some $d \equiv 1 \bmod m$ with $\eta_d = \alpha'_1$ and $\eta_{d-1} = 2\alpha'_0$. All finite type solutions may be obtained by integrating commuting Hamiltonian o.d.e.s on the finite-dimensional manifold $\Lambda_d \mathfrak{u}(m)$, and so are fairly well understood. By [2, Cor. 3.7], every superconformal map corresponding to a doubly-periodic Toda solution on $\mathbb{C}$, and hence every superconformal T^2 in $\mathbb{CP}^{m-1}$, is of finite type.

3.6 Spectral curves

To each superconformal map $\psi : U \to \mathbb{CP}^{m-1}$ of finite type one can associate a Riemann surface known as a *spectral curve*. There are two different definitions of spectral curve in use in the literature. Here is the first. Fix $z \in U$, and define

$$Y = \big\{(\lambda, [v]) \in \mathbb{C}^* \times \mathbb{CP}^{m-1} : \forall \eta \in \mathcal{A},\ \exists \mu \in \mathbb{C} \text{ with } \eta(\lambda, z)v = \mu v\big\}. \tag{18}$$

The *spectral curve* as defined by Ferus, Pedit, Pinkall and Sterling [9, §5] is the compactification $\tilde{Y}$ of Y in $\mathbb{CP}^1 \times \mathbb{CP}^{m-1}$. In the generic case, $\tilde{Y}$ is a compact, nonsingular Riemann surface.

However, McIntosh [24, App. A] uses a different definition. As each $\eta \in \mathcal{A}$ satisfies (13), if $\big(\lambda, [v]\big) \in Y$ then $\big(\zeta\lambda, [\Upsilon v]\big) \in Y$. So define $\nu : Y \to Y$ by

$$\nu : \big(\lambda, [v]\big) \mapsto \big(\zeta\lambda, [\Upsilon v]\big). \tag{19}$$

Then $\nu^m = 1$, and $\langle \nu \rangle$ is a free $\mathbb{Z}_m$-action on Y, which lifts to a free $\mathbb{Z}_m$-action on $\tilde{Y}$. The spectral curve used by McIntosh is equivalent to $\tilde{X} = \tilde{Y}/\langle \nu \rangle$. It is also generically a compact, nonsingular Riemann surface.

The point of (18) is that as $\mathcal{A}$ is a commutative algebra under matrix multiplication, the matrices $\eta(\lambda, z)$ can be simultaneously diagonalized for all $\eta \in \mathcal{A}$. Assume that all the common eigenspaces are 1-dimensional. Then $(\lambda, [v]) \in Y$ if $[v]$ is an eigenspace of $\eta(\lambda, z)$ for all $\eta \in \mathcal{A}$.

This suggests an alternative description of Y, using eigenvalues rather than eigenvectors. Pick $\eta \in \mathcal{A}$, and define

$$Y' = \big\{(\lambda, \mu) \in \mathbb{C}^* \times \mathbb{C} : \det\big(\mu I - \eta(\lambda, z)\big) = 0\big\}. \tag{20}$$

Define $\pi : Y \to Y'$ by $\pi : \big(\lambda, [v]\big) \mapsto (\lambda, \mu)$ when $\eta(\lambda, z)v = \mu v$. Then π is birational for generic η, and biholomorphic if η generates $\mathcal{A}$ over $\mathbb{C}[\lambda^m I, \lambda^{-m} I]$.

As a Riemann surface $\tilde{Y}$ is independent of base-point $z \in U$, but its embedding in $\mathbb{CP}^1 \times \mathbb{CP}^{m-1}$ does depend on z. However, Y' is independent of z. This is because (14) is equivalent to $\mathrm{d}(F_\lambda \eta F_\lambda^{-1}) = 0$, as $\alpha_\lambda = F_\lambda^{-1}\mathrm{d}F_\lambda$. Thus $F_\lambda \eta F_\lambda^{-1}$ is independent of z, so the eigenvalues of $\eta(\lambda, z)$ are independent of z.

Spectral curves are used by McIntosh [22, 23, 24] to give a construction of all finite type harmonic maps $\psi : \mathbb{C} \to \mathbb{CP}^{m-1}$, and hence of all nonisotropic harmonic maps $\psi : T^2 \to \mathbb{CP}^{m-1}$. Above we have only considered superconformal ψ, which are dealt with in [24]; the general case is studied in [22, 23], and is rather more complicated.

By an explicit construction, McIntosh establishes a 1-1 correspondence between finite type nonisotropic harmonic maps $\psi : \mathbb{C} \to \mathbb{CP}^{m-1}$ up to isometry, and quadruples of *spectral data* $(X, \sigma, \pi, \mathcal{L})$, where X is a Riemann surface, $\sigma : X \to X$ a real structure, $\pi : X \to \mathbb{CP}^1$ a branched cover, and $\mathcal{L}$ a holomorphic line bundle over X, all satisfying certain conditions. Understanding the set of such $(X, \sigma, \pi, \mathcal{L})$ is a fairly straightforward problem in algebraic geometry.

When ψ is doubly-periodic it pushes down to T^2, and all nonisotropic harmonic maps $\psi : T^2 \to \mathbb{CP}^{m-1}$ arise in this way. This reduces the classification of nonisotropic harmonic tori in $\mathbb{CP}^{m-1}$ to the problem of understanding the double-periodicity conditions upon $(X, \sigma, \pi, \mathcal{L})$, which will be discussed in §4.3.

4 SL cones in $\mathbb{C}^3$ and the Tzitzéica equation

Suppose N is a *special Lagrangian cone* in $\mathbb{C}^3$. Then $\Sigma = N \cap \mathcal{S}^5$ is a *minimal Legendrian surface* in $\mathcal{S}^5$, and its projection $\pi(\Sigma)$ to $\mathbb{CP}^2$ is a *minimal Lagrangian surface* in $\mathbb{CP}^2$. Thus, Σ and $\pi(\Sigma)$ are the images of *conformal harmonic maps* $\phi : S \to \mathcal{S}^5$ and $\psi : S \to \mathbb{CP}^2$, where S is a Riemann surface.

Therefore, we can apply the theory of §3 to ψ. It turns out that as $\psi(S)$ is Lagrangian, the corresponding solutions $\chi_k : U \to (0, \infty)$ of the SU(3) Toda lattice equations simplify, coming from a single function $f : U \to \mathbb{R}$ satisfying the *Tzitzéica equation*. This will be explained in §4.1.

The programme of §3.4–§3.6 can then be applied, to interpret finite type solutions of the Tzitzéica equation in terms of *spectral data*. The difference with the SU(3) Toda lattice case is that the spectral curve X acquires an extra symmetry, a holomorphic involution ρ satisfying some compatibility conditions with the other data $\sigma, \pi, \mathcal{L}$. The details can be found in Sharipov [25] and Ma and Ma [21], on which this section is based.

4.1 Derivation of the Tzitzéica equation

Suppose N is a special Lagrangian cone in $\mathbb{C}^3$. Define $\Sigma = N \cap \mathcal{S}^5$. Then Σ is a *minimal Legendrian surface* in $\mathcal{S}^5$, as N is a minimal Lagrangian 3-fold in $\mathbb{C}^3$. It has a natural metric and orientation, and so inherits the structure of a Riemann surface. Let S be Σ, regarded as an abstract Riemann surface, and $\phi : S \to \Sigma$ the inclusion map.

Then ϕ is conformal by definition, and has minimal image, so it is harmonic. Let $\pi : \mathcal{S}^5 \to \mathbb{CP}^2$ be the Hopf projection, and define $\psi : S \to \mathbb{CP}^2$ by $\psi = \pi \circ \phi$. As ϕ has Legendrian image it follows that ψ is also a conformal harmonic map, with *Lagrangian image*. Therefore we can consider the harmonic sequence (ψ_k) of ψ, as in §3.1.

Let $z = x + iy$ be a holomorphic coordinate on an open subset $U \subset S$. Then $g\bigl(\phi, \frac{\partial\phi}{\partial x}\bigr) = g\bigl(\phi, \frac{\partial\phi}{\partial y}\bigr) = 0$ as $|\phi| \equiv 1$, and $\omega\bigl(\phi, \frac{\partial\phi}{\partial x}\bigr) = \omega\bigl(\phi, \frac{\partial\phi}{\partial y}\bigr) = 0$ as N is Lagrangian. Hence, ϕ is complex orthogonal to $\frac{\partial\phi}{\partial\bar z}$, so ϕ is a *holomorphic* section of L_0.

Thus, by §3.1, there is a unique sequence (ϕ_k) with $\phi_0 = \phi$ satisfying equation (2). As $|\phi_0| \equiv 1$, we see that

$$\phi_0 = \phi, \quad \phi_1 = \frac{\partial\phi}{\partial z} \quad \text{and} \quad \phi_{-1} = -\Bigl|\frac{\partial\phi}{\partial\bar z}\Bigr|^{-2} \cdot \frac{\partial\phi}{\partial\bar z}. \tag{21}$$

Suppose now that $\psi : S \to \mathbb{CP}^2$ is *superconformal*, and that z is a *special* holomorphic coordinate. Then ϕ_k exists for all $k \in \mathbb{Z}$ and $\phi_{k+3} = \phi_k$, so that (21) determines ϕ_k for all k. Also, the functions $\chi_k = |\phi_k|^2$ satisfy the *Toda lattice equations* (3)–(4) for $m = 3$.

Now in this case the Toda lattice equations simplify. For $\chi_0 \equiv 1$ as $|\phi_0| \equiv 1$, and so $\chi_2 \equiv \chi_1^{-1}$ as $\chi_0\chi_1\chi_2 \equiv 1$. Defining $f = \log\chi_1$ gives $\chi_{3k} = 1$, $\chi_{3k+1} = \mathrm{e}^f$ and $\chi_{3k-1} = \mathrm{e}^{-f}$ for $f : U \to \mathbb{R}$, and then (4) reduces to the single equation

$$\frac{\partial^2 f}{\partial z\partial\bar z} = \mathrm{e}^{-2f} - \mathrm{e}^f. \tag{22}$$

This is the elliptic version of the *Tzitzéica equation*. The corresponding hyperbolic equation first arose in 1910 in a study by Georges Tzitzéica [26] of a class of surfaces in $\mathbb{RP}^3$ now known as *affine spheres*. The equation was rediscovered in a solitonic context by Bullough and Dodd [4], and amongst mathematical physicists is often known as the Bullough–Dodd equation.

We have shown that superconformal harmonic maps $\psi : S \to \mathbb{CP}^2$ coming from special Lagrangian cones in $\mathbb{C}^3$ are related to solutions of the Tzitzéica equation (22), in the same way that general superconformal harmonic maps $\psi : S \to \mathbb{CP}^{m-1}$ are related to solutions of the $\mathrm{SU}(m)$ Toda lattice equations. The converse also applies, in that starting with a solution of (22) one can reconstruct a special Lagrangian cone in $\mathbb{C}^3$ using the Toda frame method of §3.3.

4.2 Spectral data for the Tzitzéica equation

Suppose f is a solution of the Tzitzéica equation. Then as in §4.1 we get a solution (χ_k) of the $\mathrm{SU}(3)$ Toda lattice equations. If (χ_k) is of finite type then

as in §3.6 it has a set of *spectral data* $(X, \sigma, \pi, \mathcal{L})$. But because the (χ_k) come from a solution of the Tzitzéica equation and so have a simplified structure, the spectral data $(X, \sigma, \pi, \mathcal{L})$ has an extra symmetry.

It turns out that this symmetry is a *holomorphic involution* $\rho : X \to X$. It commutes with σ, has the property that if $\pi(x) = [1, \lambda]$ then $\pi \circ \rho(x) = [1, -\lambda]$ for $x \in X$ and $\lambda \in \mathbb{C}$, and lifts to a holomorphic involution of the line bundle $\mathcal{L}$.

Thus, just as there is a correspondence between finite type solutions of the $\mathrm{SU}(m)$ Toda lattice equations and quadruples of spectral data $(X, \sigma, \pi, \mathcal{L})$ satisfying certain conditions, there is also a correspondence between solutions of the Tzitzéica equation and quintuples of spectral data $(X, \rho, \sigma, \pi, \mathcal{L})$ satisfying certain conditions.

This construction is used in two papers by Sharipov [25] and Ma and Ma [21]. Sharipov considers 'complex normal' surfaces in $\mathcal{S}^5$, which in our terminology are just Legendrian surfaces. He shows that minimal Legendrian tori correspond to solutions of the Tzitzéica equation (22), gives the spectral data for finite type solutions, and sketches how to write the maps $\phi : \mathbb{R}^2 \to \mathcal{S}^5$ in terms of Prym theta functions.

The paper by Ma and Ma is very similar. They consider 'totally real' surfaces in $\mathbb{CP}^2$, which in our terminology are just Lagrangian surfaces. They show that minimal Lagrangian tori correspond to solutions of (22), give the spectral data, and write the maps $\psi : \mathbb{R}^2 \to \mathbb{CP}^2$ in terms of Prym theta functions. The principal difference is that Ma and Ma give more proofs, more detail, and more explicit formulae.

4.3 Parameter counts for minimal tori in $\mathbb{CP}^2$

We shall now use the integrable systems set-up above to give parameter counts for the families of minimal tori in $\mathbb{CP}^2$ and of minimal Lagrangian tori in $\mathbb{CP}^2$ (equivalently, of special Lagrangian T^2-cones in $\mathbb{C}^3$ up to isomorphism).

Similar parameter counts for case of minimal tori in $\mathbb{CP}^{m-1}$ are given by McIntosh in [23, p. 516] and [24, Th. 5], and we follow his method, modifying it in the obvious way for the minimal Lagrangian case by requiring invariance under the holomorphic involution ρ. Each set of spectral data corresponds up to isometries of $\mathbb{CP}^2$ with a unique finite type harmonic map $\psi : \mathbb{R}^2 \to \mathbb{CP}^2$. We shall count the number of free parameters in the spectral data, and the number of restrictions for ψ to be doubly-periodic.

First consider general minimal tori in $\mathbb{CP}^2$. The spectral data for this is a quadruple $(X, \sigma, \pi, \mathcal{L})$. We suppose that X is a nonsingular Riemann surface of genus $p \geqslant 2$. (The case $p \leqslant 1$ is dealt with in [24, §5].) Now $\pi : X \to \mathbb{CP}^1$ is a 3-fold branched cover, which we can regard as a *meromorphic function*, identifying $\mathbb{CP}^1$ with $\mathbb{C} \cup \{\infty\}$. It is required that $\pi^{-1}(0)$ and $\pi^{-1}(\infty)$ should be single points, and thus triple branch points.

When $(X, \sigma, \pi, \mathcal{L})$ is generic, all other branch points of π will be double branch points, and there will be $2p$ of them by elementary topology. Let $\lambda_1, \ldots, \lambda_{2p}$ be the images of these branch points in $\mathbb{C} \setminus \{0\}$. Then the λ_k are distinct, and it is required [23, Prop. 1] that no λ_k lies on the unit circle.

Now σ acts on X, and if $\pi(x) = \lambda$ then $\pi \circ \sigma(x) = 1/\bar{\lambda}$. Clearly σ must take double branch points to double branch points, and so the set $\{\lambda_1, \ldots, \lambda_{2p}\}$ is closed under $\lambda \mapsto 1/\bar{\lambda}$. As no λ_k lies on the unit circle this swaps the λ_k in pairs. Order the λ_k so that $|\lambda_k| < 1$ and $\lambda_{k+p} = 1/\bar{\lambda}_k$ for $k = 1, \ldots, p$.

The triple (X, σ, π) depends on $\lambda_1, \ldots, \lambda_p$ and discrete data. Thus there are $2p$ real parameters in (X, σ, π). The set of σ-invariant line bundles $\mathcal{L}$ has dimension p, but $\mathcal{L}$ depends on a choice of base point in $\mathbb{R}^2$, so factoring out by translations in $\mathbb{R}^2$ shows that the choice of $\mathcal{L}$ really represents $p-2$ degrees of freedom.

From [24, §5], the double-periodicity conditions for $\psi : \mathbb{R}^2 \to \mathbb{CP}^2$ depend only on (X, σ, π), being independent of $\mathcal{L}$. The condition for the Toda solution to be doubly-periodic is $2p - 4$ rationality conditions, and for ψ to be doubly periodic in $\mathbb{CP}^2$ is another 4 rationality conditions.

Effectively, this means that the moduli space $\mathcal{M}_p$ of data (X, σ, π) has dimension $2p$, and to be doubly-periodic requires that $2p$ real functions $f_1, \ldots, f_{2p}$ on this moduli space have rational values. If $f_1, \ldots, f_{2p}$ are locally transverse, then the set of (X, σ, π) giving doubly-periodic ψ will be countable and dense in $\mathcal{M}_p$. Having fixed (X, σ, π) there are $p - 2$ degrees of freedom to choose $\mathcal{L}$, all of which give doubly-periodic ψ.

Next we do a similar parameter count for minimal Lagrangian tori. To do this we have to include the holomorphic involution $\rho : X \to X$ as in §4.2. This has the property that if $\pi(x) = \lambda$ then $\pi \circ \rho(x) = -\lambda$. Clearly, ρ takes branch points to branch points, so the set $\{\lambda_1, \ldots, \lambda_{2p}\}$ is invariant under $\lambda \mapsto -\lambda$. It follows that p is even, say $p = 2d$, and we can order the λ_k such that $\lambda_{d+k} = -\lambda_k$ for $k = 1, \ldots, d$.

Thus, $\lambda_1, \ldots, \lambda_{4d}$ are determined by $\lambda_1, \ldots, \lambda_d$, so there are $2d$ real parameters in (X, ρ, σ, π). Suppose that $d \geqslant 2$, so that $p \geqslant 4$. (If $d = 1$ the solutions have an $\mathbb{R}$ symmetry group, and the parameter count is slightly different.) The condition for the Tzitzéica solution to be doubly-periodic is $2d - 4$ rationality conditions, and for ψ to be doubly-periodic in $\mathbb{CP}^2$ is another 4 rationality conditions. As L must be invariant under ρ and σ, it has $d - 2$ degrees of freedom.

Effectively, this means that the moduli space $\mathcal{M}'_{2d}$ of data (X, ρ, σ, π) has dimension $2d$, and to be doubly-periodic requires that $2d$ real functions $f_1, \ldots, f_{2d}$ on this moduli space have rational values. Having fixed (X, σ, π) there are $d-2$ degrees of freedom to choose $\mathcal{L}$, all of which give doubly-periodic ψ.

Here are our conclusions in brief:

- Up to isometries of $\mathbb{CP}^2$, we expect the family of minimal tori in $\mathbb{CP}^2$ with spectral curve of genus $p \geqslant 2$ to depend on $2p$ rational numbers and $p-2$ real numbers.

- Up to isometries of $\mathbb{CP}^2$, we expect the family of minimal Lagrangian tori in $\mathbb{CP}^2$ with spectral curve of genus $2d \geqslant 4$ to depend on $2d$ rational numbers and $d - 2$ real numbers.

As the double-periodicity conditions for Lagrangian $\psi : \mathbb{R}^2 \to \mathbb{CP}^2$ are equivalent to those for its Legendrian lift $\phi : \mathbb{R}^2 \to \mathcal{S}^5$, the second parameter count also gives the answer for the families of minimal Legendrian tori in $\mathcal{S}^5$ up to transformations in U(3), and of special Lagrangian T^2-cones in $\mathbb{C}^3$ up to transformations in SU(3).

The moral for special Lagrangian geometry is that one should expect very large numbers of SL T^2-cones in $\mathbb{C}^3$, which can even exist in continuous families up to isomorphisms. These provide many local models for singularities of SL 3-folds in Calabi–Yau 3-folds.

5 A family of special Lagrangian cones in $\mathbb{C}^3$

In [14, §8] and [15, §6] the author gave two constructions of countable families of special Lagrangian T^2-cones in $\mathbb{C}^3$, the first using U(1)-invariance, and the second by evolving a 1-parameter family of quadric cones in Lagrangian planes. Both constructions are related to work of other authors.

In particular, the section on U(1)-invariant SL cones in [14] essentially repeats the work of Castro and Urbano [7] on U(1)-invariant minimal tori in $\mathbb{CP}^2$, and was also discovered independently by Haskins [13]. The 'evolving quadrics' construction of [15] generalizes examples of Lawlor [20], and some of the examples it produces were also studied by Bryant [5, §3.5] from a different point of view. For more details, see [14, 15].

Motivated by these we shall now construct a more general family of special Lagrangian cones in $\mathbb{C}^3$ which includes those of [14, 15] as special cases. These some from a family of explicit conformal harmonic maps $\phi : \mathbb{R}^2 \to \mathcal{S}^5$ with Legendrian image, which will be analyzed from the integrable systems point of view in §6.

5.1 Constructing the family

Here is our main result.

Theorem 5.1 *Let $\beta_1, \beta_2, \beta_3$ and $\gamma_1, \gamma_2, \gamma_3$ be real numbers with not all β_j and not all γ_j zero, such that*

$$\beta_1 + \beta_2 + \beta_3 = 0, \quad \gamma_1 + \gamma_2 + \gamma_3 = 0 \quad \textit{and} \quad \beta_1\gamma_1 + \beta_2\gamma_2 + \beta_3\gamma_3 = 0. \tag{23}$$

Suppose $y_1, y_2, y_3 : \mathbb{R} \to \mathbb{C}$ and $v : \mathbb{R} \to \mathbb{R}$ are functions of s, and $z_1, z_2, z_3 : \mathbb{R} \to \mathbb{C}$ and $w : \mathbb{R} \to \mathbb{R}$ functions of t, satisfying

$$\frac{\mathrm{d}y_1}{\mathrm{d}s} = \beta_1\,\overline{y_2 y_3}, \qquad \frac{\mathrm{d}y_2}{\mathrm{d}s} = \beta_2\,\overline{y_3 y_1}, \qquad \frac{\mathrm{d}y_3}{\mathrm{d}s} = \beta_3\,\overline{y_1 y_2}, \tag{24}$$

$$\frac{\mathrm{d}z_1}{\mathrm{d}t} = \gamma_1\,\overline{z_2 z_3}, \qquad \frac{\mathrm{d}z_2}{\mathrm{d}t} = \gamma_2\,\overline{z_3 z_1}, \qquad \frac{\mathrm{d}z_3}{\mathrm{d}t} = \gamma_3\,\overline{z_1 z_2}, \tag{25}$$

$$|y_1|^2 = \beta_1 v + 1, \qquad |y_2|^2 = \beta_2 v + 1, \qquad |y_3|^2 = \beta_3 v + 1, \tag{26}$$

$$|z_1|^2 = \gamma_1 w + 1, \qquad |z_2|^2 = \gamma_2 w + 1, \qquad |z_3|^2 = \gamma_3 w + 1. \tag{27}$$

If (24)–(25) *hold for all* s, t *and* (26)–(27) *hold for* $s = t = 0$*, then* (26)–(27) *hold for all* s, t*, for some functions* v, w*. Define* $\Phi : \mathbb{R}^3 \to \mathbb{C}^3$ *by*

$$\Phi : (r, s, t) \mapsto \tfrac{1}{\sqrt{3}}\big(ry_1(s)z_1(t), ry_2(s)z_2(t), ry_3(s)z_3(t)\big). \tag{28}$$

Define a subset N *of* $\mathbb{C}^3$ *by*

$$N = \big\{\Phi(r, s, t) : r, s, t \in \mathbb{R}\big\}. \tag{29}$$

Then N *is a special Lagrangian cone in* $\mathbb{C}^3$*.*

Proof. Suppose (24) holds for all s, and (26) for $s = 0$. From (24) we deduce that $\frac{\mathrm{d}}{\mathrm{d}s}\big(|y_j|^2\big) = 2\beta_j \operatorname{Re}(y_1y_2y_3)$ for $j = 1, 2, 3$. Comparing this with (26) shows that $v(s)$ should satisfy $\frac{\mathrm{d}v}{\mathrm{d}s} = 2\operatorname{Re}(y_1y_2y_3)$. Therefore, setting $v(s) = v(0) + 2\int_0^s \operatorname{Re}\big(y_1(u)y_2(u)y_3(u)\big)\mathrm{d}u$ shows that (26) holds for all s. Similarly, if (25) holds for all t and (27) holds for $t = 0$, then it holds for all t. This proves the first part of the theorem.

For the second part, we must show that N is special Lagrangian wherever it is nonsingular, that is, wherever Φ is an immersion. Now Φ is an immersion at (r, s, t) when $\frac{\partial\Phi}{\partial r}, \frac{\partial\Phi}{\partial s}, \frac{\partial\Phi}{\partial t}$ are linearly independent, and then $T_{\Phi(r,s,t)}N = \big\langle\frac{\partial\Phi}{\partial r}, \frac{\partial\Phi}{\partial s}, \frac{\partial\Phi}{\partial t}\big\rangle_{\mathbb{R}}$. Thus we must show that $T_{\Phi(r,s,t)}N$ is an SL 3-plane $\mathbb{R}^3$ in $\mathbb{C}^3$ for all (r, s, t) for which $\frac{\partial\Phi}{\partial r}, \frac{\partial\Phi}{\partial s}, \frac{\partial\Phi}{\partial t}$ are linearly independent. By Proposition 2.3, this holds if and only if

$$\omega\Big(\frac{\partial\Phi}{\partial r}, \frac{\partial\Phi}{\partial s}\Big) \equiv \omega\Big(\frac{\partial\Phi}{\partial r}, \frac{\partial\Phi}{\partial t}\Big) \equiv \omega\Big(\frac{\partial\Phi}{\partial s}, \frac{\partial\Phi}{\partial t}\Big) \equiv 0 \tag{30}$$

$$\text{and} \qquad \operatorname{Im}\Omega\Big(\frac{\partial\Phi}{\partial r}, \frac{\partial\Phi}{\partial s}, \frac{\partial\Phi}{\partial t}\Big) \equiv 0. \tag{31}$$

Using equations (24), (25) and (28) we find that

$$\frac{\partial\Phi}{\partial r} = \tfrac{1}{\sqrt{3}}\big(y_1z_1, y_2z_2, y_3z_3\big), \tag{32}$$

$$\frac{\partial\Phi}{\partial s} = \tfrac{1}{\sqrt{3}}\big(r\beta_1\,\overline{y_2y_3}z_1, r\beta_2\,\overline{y_3y_1}z_2, r\beta_3\,\overline{y_1y_2}z_3\big), \tag{33}$$

$$\frac{\partial\Phi}{\partial t} = \tfrac{1}{\sqrt{3}}\big(r\gamma_1y_1\,\overline{z_2z_3}, r\gamma_2y_2\,\overline{z_3z_1}, r\gamma_3y_3\,\overline{z_1z_2}\big). \tag{34}$$

From (1) we deduce that $\omega\big((a_1, a_2, a_3), (b_1, b_2, b_3)\big) = \operatorname{Im}(a_1\bar b_1 + a_2\bar b_2 + a_3\bar b_3)$. Thus from (32) and (33) we have

$$\begin{aligned}
\omega\big(\tfrac{\partial\Phi}{\partial r}, \tfrac{\partial\Phi}{\partial s}\big) &= \tfrac{1}{3}r\operatorname{Im}(y_1y_2y_3)\big(\beta_1|z_1|^2 + \beta_2|z_2|^2 + \beta_3|z_3|^2\big)\\
&= \tfrac{1}{3}r\operatorname{Im}(y_1y_2y_3)\big(\beta_1(\gamma_1w + 1) + \beta_2(\gamma_2w + 1) + \beta_3(\gamma_3w + 1)\big)\\
&= \tfrac{1}{3}r\operatorname{Im}(y_1y_2y_3)\big(\beta_1 + \beta_2 + \beta_3 + w(\beta_1\gamma_1 + \beta_2\gamma_2 + \beta_3\gamma_3)\big) = 0,
\end{aligned}$$

using (27) in the second line and (23) in the third. This proves the first equation of (30). The second follows in the same way, and the third from

$$\omega\big(\tfrac{\partial\Phi}{\partial s}, \tfrac{\partial\Phi}{\partial t}\big) = \tfrac{1}{3}r^2\operatorname{Im}(\,\overline{y_1y_2y_3}z_1z_2z_3)(\beta_1\gamma_1 + \beta_2\gamma_2 + \beta_3\gamma_3) = 0,$$

using (23), (33) and (34).

To prove (31), observe that

$$\Omega\Big(\frac{\partial\Phi}{\partial r},\frac{\partial\Phi}{\partial s},\frac{\partial\Phi}{\partial t}\Big)=\Big|\frac{\partial\Phi}{\partial r}\;\frac{\partial\Phi}{\partial s}\;\frac{\partial\Phi}{\partial t}\Big|=\frac{1}{3\sqrt{3}}\begin{vmatrix} y_1z_1 & r\beta_1\,\overline{y_2y_3}z_1 & r\gamma_1y_1\,\overline{z_2z_3} \\ y_2z_2 & r\beta_2\,\overline{y_3y_1}z_2 & r\gamma_2y_2\,\overline{z_3z_1} \\ y_3z_3 & r\beta_3\,\overline{y_1y_2}z_3 & r\gamma_3y_3\,\overline{z_1z_2}\end{vmatrix}$$

$$=\tfrac{1}{3\sqrt{3}}r^2\big(\beta_2|y_3y_1|^2\gamma_3|z_1z_2|^2+\beta_3|y_1y_2|^2\gamma_1|z_2z_3|^2+\beta_1|y_2y_3|^2\gamma_2|z_3z_1|^2$$
$$-\beta_3|y_1y_2|^2\gamma_2|z_3z_1|^2-\beta_1|y_2y_3|^2\gamma_3|z_1z_2|^2-\beta_2|y_3y_1|^2\gamma_1|z_2z_3|^2\big),$$

where in the first line the terms $|\dots|$ are determinants of complex 3×3 matrices, and $\frac{\partial\Phi}{\partial r},\frac{\partial\Phi}{\partial s},\frac{\partial\Phi}{\partial t}$ are regarded are complex column matrices. As every term in the second line is real, $\Omega\big(\frac{\partial\Phi}{\partial r},\frac{\partial\Phi}{\partial s},\frac{\partial\Phi}{\partial t}\big)$ is real. Thus $\operatorname{Im}\Omega\big(\frac{\partial\Phi}{\partial r},\frac{\partial\Phi}{\partial s},\frac{\partial\Phi}{\partial t}\big)=0$, proving (31). □

5.2 Explicit solution of the o.d.e.s (24) and (25)

As in [14, §8] and [15, §6], we can simplify the solutions of (24) and (25). To do this, note first that (24) implies that

$$\tfrac{\mathrm{d}}{\mathrm{d}s}(y_1y_2y_3)=\beta_1|y_2y_3|^2+\beta_2|y_3y_1|^2+\beta_3|y_1y_2|^2.$$

As the right hand side is real, we have $\operatorname{Im}(y_1y_2y_3)\equiv B$ for some $B\in\mathbb{R}$.

Now $|y_1y_2y_3|^2=(\beta_1v+1)(\beta_2v+1)(\beta_3v+1)$, so this gives

$$\big|\operatorname{Re}(y_1y_2y_3)\big|^2=(\beta_1v+1)(\beta_2v+1)(\beta_3v+1)-B^2.$$

However, $\frac{\mathrm{d}v}{\mathrm{d}s}=2\operatorname{Re}(y_1y_2y_3)$, as in the proof of Theorem 5.1, and so v satisfies the o.d.e.

$$\Big(\frac{\mathrm{d}v}{\mathrm{d}s}\Big)^2=4\big((\beta_1v+1)(\beta_2v+1)(\beta_3v+1)-B^2\big).$$

Since $|y_j|^2=\beta_jv+1$ by (26) we may write $y_j(s)=\mathrm{e}^{i\delta_j(s)}\sqrt{\beta_jv(s)+1}$ for $j=1,2,3$, for real functions $\delta_1,\delta_2,\delta_3$. In this way we prove:

Proposition 5.2 *In the situation of Theorem 5.1 the functions y_1,y_2,y_3 may be written $y_j(s)=\mathrm{e}^{i\delta_j(s)}\sqrt{\beta_jv(s)+1}$, for $v,\delta_1,\delta_2,\delta_3:\mathbb{R}\rightarrow\mathbb{R}$. Define $Q(v)=(\beta_1v+1)(\beta_2v+1)(\beta_3v+1)$ and $\delta=\delta_1+\delta_2+\delta_3$. Then (26) holds automatically, and (24) is equivalent to*

$$\Big(\frac{\mathrm{d}v}{\mathrm{d}s}\Big)^2=4\big(Q(v)-B^2\big)\quad\textit{and}\quad\frac{\mathrm{d}\delta_j}{\mathrm{d}s}=-\frac{\beta_jB}{\beta_jv+1}\tag{35}$$

for $j=1,2,3$, where $\operatorname{Im}(y_1y_2y_3)\equiv Q(v)^{1/2}\sin\delta\equiv B$ for some $B\in[-1,1]$.

Here is the corresponding result for z_1,z_2,z_3.

Proposition 5.3 *In the situation of Theorem 5.1 the functions z_1, z_2, z_3 may be written $z_j(t) = \mathrm{e}^{i\epsilon_j(t)}\sqrt{\gamma_j w(t)+1}$, for $w, \epsilon_1, \epsilon_2, \epsilon_3 : \mathbb{R} \to \mathbb{R}$. Define $R(w) = (\gamma_1 w+1)(\gamma_2 w+1)(\gamma_3 w+1)$ and $\epsilon = \epsilon_1+\epsilon_2+\epsilon_3$. Then* (27) *holds automatically, and* (25) *is equivalent to*

$$\left(\frac{\mathrm{d}w}{\mathrm{d}t}\right)^2 = 4\bigl(R(w) - C^2\bigr) \quad \textit{and} \quad \frac{\mathrm{d}\epsilon_j}{\mathrm{d}t} = -\frac{\gamma_j C}{\gamma_j w + 1} \tag{36}$$

for $j = 1, 2, 3$, where $\mathrm{Im}(z_1 z_2 z_3) \equiv R(w)^{1/2} \sin\epsilon \equiv C$ for some $C \in [-1, 1]$.

As in [14, §8.2], the o.d.e.s for v and w in (35) and (36) can be solved entirely explicitly in terms of the *Jacobi elliptic functions*. Then δ_j and ϵ_j can also be given explicitly, in terms of integrals involving the Jacobi elliptic functions, and so the solutions y_j, z_j of (24) and (25) are known explicitly in terms of elliptic functions. We shall not give these solutions here.

5.3 Conformal harmonic maps to $\mathcal{S}^5$ and $\mathbb{CP}^2$

As in §4.1, a special Lagrangian cone in $\mathbb{C}^3$ induces conformal harmonic maps $\phi : S \to \mathcal{S}^5$ and $\psi : S \to \mathbb{CP}^2$ from a Riemann surface S. We shall now write these out explicitly for the SL cones of Theorem 5.1. For convenience, we begin by choosing a normalization for the constants $\beta_1, \beta_2, \beta_3$ and $\gamma_1, \gamma_2, \gamma_3$.

Regarding $\boldsymbol{\beta} = (\beta_1, \beta_2, \beta_3)$ and $\boldsymbol{\gamma} = (\gamma_1, \gamma_2, \gamma_3)$ as vectors in $\mathbb{R}^3$, the conditions on $\boldsymbol{\beta}, \boldsymbol{\gamma}$ in Theorem 5.1 are that $\boldsymbol{\beta}$ and $\boldsymbol{\gamma}$ should be nonzero, and that $\boldsymbol{\beta}, \boldsymbol{\gamma}$ and $(1, 1, 1)$ should be orthogonal. However, multiplying $\boldsymbol{\beta}$ or $\boldsymbol{\gamma}$ by a nonzero constant has no effect on the set of special Lagrangian cones constructed in Theorem 5.1.

To see this, let $\beta_j, \gamma_j, y_j, z_j, v$ and w satisfy the conditions of the theorem, let $\sigma, \tau \in \mathbb{R}$ be nonzero, and define

$$\begin{aligned} &\beta'_j = \sigma\beta_j, \quad y'_j(s) = y_j(\sigma s) \quad \text{for } j = 1, 2, 3, \text{ and} \quad v'(s) = \sigma^{-1} v(\sigma s), \\ &\gamma'_j = \tau\gamma_j, \quad z'_j(t) = z_j(\tau t) \quad \text{for } j = 1, 2, 3, \text{ and} \quad w'(t) = \tau^{-1} w(\tau t). \end{aligned} \tag{37}$$

Then it is easy to show that $\beta'_j, \gamma'_j, y'_j, z'_j, v'$ and w' also satisfy the conditions of the theorem, yielding $\Phi' : \mathbb{R}^3 \to \mathbb{C}^3$ with $\Phi'(r, s, t) = \Phi(r, \sigma s, \tau t)$, so that the images of Φ' and Φ are the same special Lagrangian cone.

Therefore, we are free to rescale $\boldsymbol{\beta}$ and $\boldsymbol{\gamma}$ without changing the resulting set of SL cones. Fix $|\boldsymbol{\beta}| = |\boldsymbol{\gamma}| = 1$. Then $\boldsymbol{\beta}$ and $\boldsymbol{\gamma}$ lie on the unit circle in the plane $x_1 + x_2 + x_3 = 0$ in $\mathbb{R}^3$ and are orthogonal, so we may write

$$\begin{aligned} &\boldsymbol{\beta} = \cos\theta \cdot \tfrac{1}{\sqrt{2}}(1, -1, 0) + \sin\theta \cdot \tfrac{1}{\sqrt{6}}(-1, -1, 2) \\ \text{and} \quad &\boldsymbol{\gamma} = \cos\theta \cdot \tfrac{1}{\sqrt{6}}(-1, -1, 2) - \sin\theta \cdot \tfrac{1}{\sqrt{2}}(1, -1, 0) \end{aligned}$$

for some $\theta \in [0, 2\pi)$. (Here $\boldsymbol{\beta}$ determines $\boldsymbol{\gamma}$ up to sign, which we have chosen arbitrarily.) We shall show that with these choices, the map $(s, t) \mapsto \Phi(1, s, t)$ is conformal.

Theorem 5.4 *Fix $\theta \in [0, 2\pi)$, and define*

$$\beta_1 = \tfrac{1}{\sqrt{2}}\cos\theta - \tfrac{1}{\sqrt{6}}\sin\theta, \qquad \beta_2 = -\tfrac{1}{\sqrt{2}}\cos\theta - \tfrac{1}{\sqrt{6}}\sin\theta, \qquad \beta_3 = \tfrac{2}{\sqrt{6}}\sin\theta, \tag{38}$$

$$\gamma_1 = -\tfrac{1}{\sqrt{6}}\cos\theta - \tfrac{1}{\sqrt{2}}\sin\theta, \qquad \gamma_2 = -\tfrac{1}{\sqrt{6}}\cos\theta + \tfrac{1}{\sqrt{2}}\sin\theta, \qquad \gamma_3 = \tfrac{2}{\sqrt{6}}\cos\theta. \tag{39}$$

In the situation of Theorem 5.1, with these values of β_j and γ_j, we have $|\Phi(r,s,t)|^2 = r^2$ and $\frac{\partial\Phi}{\partial r}$, $\frac{\partial\Phi}{\partial s}$ and $\frac{\partial\Phi}{\partial t}$ are orthogonal with $|\frac{\partial\Phi}{\partial r}|^2 = 1$ and

$$\bigl|\tfrac{\partial\Phi}{\partial s}\bigr|^2 = \bigl|\tfrac{\partial\Phi}{\partial t}\bigr|^2 = 2r^2\bigl(a + bv(s) + cw(t)\bigr), \tag{40}$$

$$\textit{where} \quad a = \tfrac{1}{6}(\beta_1^2 + \beta_2^2 + \beta_3^2) = \tfrac{1}{6}(\gamma_1^2 + \gamma_2^2 + \gamma_3^2) = \tfrac{1}{6}, \tag{41}$$

$$b = -\tfrac{1}{6}(\beta_1^3 + \beta_2^3 + \beta_3^3) = \tfrac{1}{6}(\beta_1\gamma_1^2 + \beta_2\gamma_2^2 + \beta_3\gamma_3^2) = -\tfrac{1}{2}\beta_1\beta_2\beta_3, \tag{42}$$

$$\textit{and} \quad c = \tfrac{1}{6}(\beta_1^2\gamma_1 + \beta_2^2\gamma_2 + \beta_3^2\gamma_3) = -\tfrac{1}{6}(\gamma_1^3 + \gamma_2^3 + \gamma_3^3) = -\tfrac{1}{2}\gamma_1\gamma_2\gamma_3. \tag{43}$$

The maps $\phi : \mathbb{R}^2 \to \mathcal{S}^5$ and $\psi : \mathbb{R}^2 \to \mathbb{CP}^2$ defined by $\phi : (s,t) \mapsto \Phi(1,s,t)$ and $\psi : (s,t) \mapsto [\Phi(1,s,t)]$ are both conformal harmonic maps.

Proof. For the first part, by (23) and (26)–(28) we have

$$\begin{aligned}
\bigl|\Phi(r,s,t)\bigr|^2 &= \tfrac{1}{3}r^2\bigl(|y_1|^2|z_1|^2 + |y_2|^2|z_2|^2 + |y_3|^2|z_3|^2\bigr) \\
&= \tfrac{1}{3}r^2\bigl((\beta_1 v+1)(\gamma_1 w+1) + (\beta_2 v+1)(\gamma_2 w+1) + (\beta_3 v+1)(\gamma_3 w+1)\bigr) \\
&= \tfrac{1}{3}r^2\bigl(3 + (\beta_1+\beta_2+\beta_3)v + (\gamma_1+\gamma_2+\gamma_3)w + (\beta_1\gamma_1+\beta_2\gamma_2+\beta_3\gamma_3)vw\bigr) \\
&= r^2.
\end{aligned}$$

The equation $|\frac{\partial\Phi}{\partial r}|^2 = 1$ follows in the same way.

To prove $\frac{\partial\Phi}{\partial r}, \frac{\partial\Phi}{\partial s}, \frac{\partial\Phi}{\partial t}$ are orthogonal we use (32)–(34) and the formula $g\bigl((a_1,a_2,a_3),(b_1,b_2,b_3)\bigr) = \mathrm{Re}\bigl(a_1\bar b_1 + a_2\bar b_2 + a_3\bar b_3\bigr)$. Thus we have

$$\begin{aligned}
g\bigl(\tfrac{\partial\Phi}{\partial r}, \tfrac{\partial\Phi}{\partial s}\bigr) &= \tfrac{1}{3}r\,\mathrm{Re}(y_1y_2y_3)\bigl(\beta_1|z_1|^2 + \beta_2|z_2|^2 + \beta_3|z_3|^2\bigr) \\
&= \tfrac{1}{3}r\,\mathrm{Re}(y_1y_2y_3)\bigl(\beta_1(\gamma_1 w+1) + \beta_2(\gamma_2 w+1) + \beta_3(\gamma_3 w+1)\bigr) \\
&= \tfrac{1}{3}r\,\mathrm{Re}(y_1y_2y_3)\bigl(\beta_1+\beta_2+\beta_3 + w(\beta_1\gamma_1+\beta_2\gamma_2+\beta_3\gamma_3)\bigr) = 0,
\end{aligned}$$

using (27) in the second line and (23) in the third. In the same way we show that $g\bigl(\frac{\partial\Phi}{\partial r}, \frac{\partial\Phi}{\partial t}\bigr) = g\bigl(\frac{\partial\Phi}{\partial s}, \frac{\partial\Phi}{\partial t}\bigr) = 0$, and so $\frac{\partial\Phi}{\partial r}, \frac{\partial\Phi}{\partial s}, \frac{\partial\Phi}{\partial t}$ are orthogonal.

Using equations (26), (27) and (33) we obtain

$$\begin{aligned}
\bigl|\tfrac{\partial\Phi}{\partial s}\bigr|^2 &= \tfrac{1}{3}r^2\bigl[\beta_1^2|y_2|^2|y_3|^2|z_1|^2 + \beta_2^2|y_3|^2|y_1|^2|z_2|^2 + \beta_3^2|y_1|^2|y_2|^2|z_3|^2\bigr] \\
&= \tfrac{1}{3}r^2\bigl[\beta_1^2(\beta_2 v+1)(\beta_3 v+1)(\gamma_1 w+1) + \beta_2^2(\beta_3 v+1)(\beta_1 v+1)(\gamma_2 w+1) \\
&\qquad + \beta_3^2(\beta_1 v+1)(\beta_2 v+1)(\gamma_3 w+1)\bigr] \\
&= \tfrac{1}{3}r^2\bigl[(\beta_1^2+\beta_2^2+\beta_3^2) + v\bigl(\beta_1^2(\beta_2+\beta_3) + \beta_2^2(\beta_3+\beta_1) + \beta_3^2(\beta_1+\beta_2)\bigr) \\
&\quad + w(\beta_1^2\gamma_1+\beta_2^2\gamma_2+\beta_3^2\gamma_3) + vw\bigl(\beta_1^2(\beta_2+\beta_3)\gamma_1 + \beta_2^2(\beta_3+\beta_1)\gamma_2 + \beta_3^2(\beta_1+\beta_2)\gamma_3\bigr) \\
&\qquad + v^2\beta_1\beta_2\beta_3(\beta_1+\beta_2+\beta_3) + v^2 w\beta_1\beta_2\beta_3(\beta_1\gamma_1+\beta_2\gamma_2+\beta_3\gamma_3)\bigr].
\end{aligned}$$

By (23), the terms in v^2 and v^2w vanish. Also, using (38) and (39) we have

$$\begin{aligned}&\beta_1^2(\beta_2+\beta_3)\gamma_1+\beta_2^2(\beta_3+\beta_1)\gamma_2+\beta_3^2(\beta_1+\beta_2)\gamma_3\\&\qquad=\tfrac{1}{\sqrt{3}}\bigl(\beta_1^2(\beta_2^2-\beta_3^2)+\beta_2^2(\beta_3^2-\beta_1^2)+\beta_3^2(\beta_1^2-\beta_2^2)\bigr)=0,\end{aligned}$$

so the term in vw vanishes. Thus, replacing $(\beta_2+\beta_3)$ by $-\beta_1$, etc., we get

$$\bigl|\tfrac{\partial\Phi}{\partial s}\bigr|^2=\tfrac{1}{3}r^2\bigl[(\beta_1^2+\beta_2^2+\beta_3^2)-v(\beta_1^3+\beta_2^3+\beta_3^3)+w(\beta_1^2\gamma_1+\beta_2^2\gamma_2+\beta_3^2\gamma_3)\bigr].$$

In the same way, we find that

$$\bigl|\tfrac{\partial\Phi}{\partial t}\bigr|^2=\tfrac{1}{3}r^2\bigl[(\gamma_1^2+\gamma_2^2+\gamma_3^2)+v(\beta_1\gamma_1^2+\beta_2\gamma_2^2+\beta_3\gamma_3^2)-w(\gamma_1^3+\gamma_2^3+\gamma_3^3)\bigr].$$

Now using (38) and (39) one can show that

$$\begin{gathered}\beta_1^2+\beta_2^2+\beta_3^2=\gamma_1^2+\gamma_2^2+\gamma_3^2,\quad -(\beta_1^3+\beta_2^3+\beta_3^3)=\beta_1\gamma_1^2+\beta_2\gamma_2^2+\beta_3\gamma_3^2=-3\beta_1\beta_2\beta_3\\ \text{and}\quad \beta_1^2\gamma_1+\beta_2^2\gamma_2+\beta_3^2\gamma_3=-(\gamma_1^3+\gamma_2^3+\gamma_3^3)=-3\gamma_1\gamma_2\gamma_3.\end{gathered}$$

The last five equations prove (40)–(43), as we want. Finally, it follows from what we have proved so far that $\phi:\mathbb{R}^2\to\mathcal{S}^5$ is a conformal map, and as its image is minimal, it is also harmonic. As ϕ has Legendrian image, ψ is also conformal and harmonic, in the usual way. □

As from §5.2 the functions y_j, z_j defining Φ are known explicitly in terms of integrals involving the Jacobi elliptic functions, we have constructed families of *explicit conformal harmonic maps* $\phi:\mathbb{R}^2\to\mathcal{S}^5$ and $\psi:\mathbb{R}^2\to\mathbb{CP}^2$.

5.4 Interesting special cases, and double periodicity

We now consider some special cases in which the y_j or z_j assume a simple form, and so explain how to recover the constructions of [14, §8] and [15, §6] from the more general construction above.

(a) Let $\kappa_1,\kappa_2,\kappa_3\in\mathbb{R}$ with $\kappa_1+\kappa_2+\kappa_3=-\pi/2$, and define $y_j=\mathrm{e}^{i(\beta_j s+\kappa_j)}$ for $j=1,2,3$. Then it is easy to see that y_1,y_2,y_3 satisfy (24) and (26), with $v\equiv 0$ and $B=-1$. The corresponding special Lagrangian cones in Theorem 5.1 are invariant under the group action

$$(z_1,z_2,z_3)\mapsto(\mathrm{e}^{i\beta_1 s}z_1,\mathrm{e}^{i\beta_2 s}z_2,\mathrm{e}^{i\beta_3 s}z_3)$$

for $s\in\mathbb{R}$, which is a U(1) subgroup of SU(3) if β_1,β_2,β_3 are relatively rational, and an $\mathbb{R}$ subgroup otherwise. In this case, Theorem 5.1 reduces to the construction of U(1)-invariant SL cones in $\mathbb{C}^3$ given in [14, §8].

In the same way, we can take $z_j=\mathrm{e}^{i(\gamma_j t+\kappa_j)}$ for $j=1,2,3$, with $w\equiv 0$ and $C=-1$, and two similar cases with $B=1$ and $C=1$, all of which give U(1)-invariant or $\mathbb{R}$-invariant SL cones in $\mathbb{C}^3$ coming from the construction of [14, §8]. (See also Castro and Urbano [7], and Haskins [13].)

(b) Take $B = 0$ in Proposition 5.2. Then (35) shows that the phases $\delta_1, \delta_2, \delta_3$ are constant, so we may as well fix them to be 0 or π, and take y_1, y_2, y_3 to be *real*. As in [15, §6.1] the y_j are given by simple formulae involving Jacobi elliptic functions (rather than integrals of Jacobi elliptic functions).

As the point (y_1, y_2, y_3) moves in $\mathbb{R}^3$ it sweeps out one of the two connected components of the curve

$$\big\{(x_1, x_2, x_3) \in \mathbb{R}^3 : x_1^2 + x_2^2 + x_3^2 = 3, \quad \gamma_1 x_1^2 + \gamma_2 x_2^2 + \gamma_3 x_3^2 = 0\big\}.$$

From this it follows that for fixed t, as r, s vary $\Phi(r, s, t)$ sweeps out a quadric cone in a Lagrangian $\mathbb{R}^3$ in $\mathbb{C}^3$. So the special Lagrangian cone N of (29) is the total space of a 1-parameter family of such quadrics, and we recover the 'evolving quadrics' construction of [15].

In the same way, if $C = 0$ in Proposition 5.3 a similar thing happens, with s and t exchanged.

(c) Set $\theta = 0$ in Theorem 5.4. Then $\beta_3 = 0$, so y_3 is constant with $|y_3| = 1$ by (24) and (26), and y_1, y_2 are linear combinations of $\mathrm{e}^{\pm is/\sqrt{2}}$. Also $\gamma_1 = \gamma_2$, so $z_2 \equiv \mathrm{e}^{i\kappa} z_1$ for some $\kappa \in \mathbb{R}$.

The corresponding SL cones in $\mathbb{C}^3$ turn out to be invariant under a U(1) subgroup of SU(3) which fixes the third coordinate in $\mathbb{C}^3$, corresponding to translation in the s variable. Thus, after a linear coordinate change in $\mathbb{C}^3$, this case reduces to a special case of the U(1)-invariant cones in part (a), but with a different parametrization.

In the same way, for each of the five other values of $\theta \in [0, 2\pi)$ for which one of $\beta_2, \beta_3, \gamma_1, \gamma_2$ and γ_3 is zero, a similar thing happens.

Next we consider when the maps $\phi : \mathbb{R}^2 \to \mathcal{S}^5$ and $\psi : \mathbb{R}^2 \to \mathbb{CP}^2$ of Theorem 5.4 are *doubly-periodic* in $\mathbb{R}^2$. Then ϕ and ψ push down to conformal harmonic maps $T^2 \to \mathcal{S}^5$ and $T^2 \to \mathbb{CP}^2$ whose images are *minimal tori* in $\mathcal{S}^5$ and $\mathbb{CP}^2$, and the special Lagrangian cone N of Theorem 5.1 is a cone on T^2. We suppose for simplicity that β_j, γ_j are normalized as in equations (38)–(39).

It turns out that in cases (a)–(c) above the double-periodicity conditions are soluble:

(a) In case (a), suppose $\beta_1, \beta_2, \beta_3$ are *relatively rational*. This happens for a countable dense set of $\theta \in [0, 2\pi)$. Then $\beta_j = n_j/S$ for $S > 0$ and n_1, n_2, n_3 coprime integers. It follows that $y_j(s+S) = y_j(s)$ for $j = 1, 2, 3$ and $s \in \mathbb{R}$, so that the y_j are periodic in s.

For double periodicity in s, t, the z_j have only to be periodic up to multiplication by $\mathrm{e}^{i\beta_j s}$ for some $s \in \mathbb{R}$. Now θ and B are already fixed, but we are free to vary the constant C in Proposition 5.3. It is shown in [14, Th. 8.5] that double periodicity holds for a countable dense subset of $C \in [-1, 1]$.

(b) In case (b) with $B=0$, y_1,y_2,y_3 are automatically periodic in s. We then need to vary the remaining data θ,C to make z_1,z_2,z_3 periodic in t.

Now w is always periodic in t, with period T, say, and the z_j transform as $z_j(t+T)=\mathrm{e}^{i\zeta_j}z_j(t)$, where $\zeta_1,\zeta_2,\zeta_3\in\mathbb{R}$ with $\zeta_1+\zeta_2+\zeta_3=0$. If $\zeta_j\in\pi\mathbb{Q}$ for $j=1,2,3$ then $n\zeta_j\in 2\pi\mathbb{Z}$ for some positive integer n, and then z_1,z_2,z_3 are periodic with period nT. So, for double periodicity we need 2 functions of θ and C to be rational. In [15, Th.s 5.9, 6.3 & 6.4] it is shown that the z_j are periodic for a countable dense set of values of (θ,C).

(c) In case (c), y_1,y_2,y_3 are automatically periodic with period $2\sqrt{2}\pi$. Also, as $z_2\equiv \mathrm{e}^{i\kappa}z_1$, the periodicity conditions for z_1,z_2,z_3 reduce to one rationality condition, rather than two. As in case (a), z_1,z_2,z_3 are periodic in t for a countable dense subset of $C\in[-1,1]$.

What about double periodicity conditions in the general case? If $|B|=1$ then v is constant and we are in case (a) above, so suppose $|B|<1$, and similarly $|C|<1$. Then v,w are automatically nonconstant and periodic in s,t, with periods S,T say, and the y_j and z_j transform as

$$y_j(s+S)=\mathrm{e}^{i\eta_j}y_j(s)\quad\text{and}\quad z_j(t+T)=\mathrm{e}^{i\zeta_j}z_j(t)$$

for some constants $\eta_j,\zeta_j\in\mathbb{R}$ with $\eta_1+\eta_2+\eta_3=\zeta_1+\zeta_2+\zeta_3=0$. The conditions for the y_j and z_j to be periodic in s and t are that $\eta_j\in\pi\mathbb{Q}$ and $\zeta_j\in\pi\mathbb{Q}$ for $j=1,2,3$ respectively.

Thus, for ϕ and ψ to be doubly-periodic we need the four functions η_1/π, η_2/π, ζ_1/π, ζ_2/π of the three variables θ,B,C to be rational. This is an *overdetermined* problem, so it seems likely that in the general case, the double periodicity conditions will have few solutions, or none. Other than parts (a)–(c) abive, the author knows of no cases in which ϕ,ψ are doubly-periodic.

We can use (40) to give a formula for the area of the minimal tori in $\mathcal{S}^5$ or $\mathbb{CP}^2$ arising from the construction above.

Proposition 5.5 *Suppose that the map $\phi:\mathbb{R}^2\to\mathcal{S}^5$ defined in Theorem 5.4 is doubly-periodic in (s,t), with image Σ, so that Σ is a minimal torus in $\mathcal{S}^5$. Let S,T be the periods of v,w in s and t, as above, and let the period lattice of $(s,t)\mapsto\Phi(1,s,t)$ in $\mathbb{R}^2$ be generated by $(a_{11}S,a_{12}T)$ and $(a_{21}S,a_{22}T)$ for integers a_{ij}. Let $N=|a_{11}a_{22}-a_{12}a_{21}|$. Then the area of Σ is*

$$\mathrm{Area}(\Sigma)=2N\Big(aST+bT\int_0^S v(s)\mathrm{d}s+cS\int_0^T w(t)\mathrm{d}t\Big).\tag{44}$$

Proof. As $\frac{\partial\Phi}{\partial s}$ and $\frac{\partial\Phi}{\partial t}$ are orthogonal, (40) implies that the area form on Σ is $2\big(a+bv(s)+cw(t)\big)\mathrm{d}s\wedge\mathrm{d}t$. Also, as the period lattice is generated by $(a_{11}S,a_{12}T)$ and $(a_{21}S,a_{22}T)$, we can divide Σ into $N=|a_{11}a_{22}-a_{12}a_{21}|$ copies of the basic rectangle $[0,S]\times[0,T]$, each of which has area $\int_0^T\int_0^S 2\big(a+bv(s)+cw(t)\big)\mathrm{d}s\mathrm{d}t$. Equation (44) follows immediately. $\square$

Observe that v, w can be written explicitly using Jacobi elliptic functions as in §5.2, and so (44) could easily be evaluated numerically in examples using a computer. This may be valuable in studying singularities of special Lagrangian 3-folds, since the area of Σ is a crude measure of how nongeneric singularities modelled on the cone on Σ are in the family of all special Lagrangian 3-folds. Also, note that the area of Σ in $\mathcal{S}^5$ is the same as the area of its image in $\mathbb{CP}^2$, as the two are isometric.

5.5 Comparison with constant mean curvature tori in $\mathbb{R}^3$

There is a strong analogy between the minimal Lagrangian tori in $\mathbb{CP}^2$ constructed above, and the examples of *constant mean curvature (CMC) tori* in $\mathbb{R}^3$ constructed by Wente [27] and Abresch [1], known as *Wente tori*. Wente proved [27] using analysis that there exist immersed CMC tori in $\mathbb{R}^3$, and so provided the first counterexamples to a conjecture of Hopf that the only compact surfaces in $\mathbb{R}^3$ with constant mean curvature are round spheres.

Motivated by Wente's construction, Abresch [1] gave explicit formulae for the Wente tori in terms of elliptic integrals. Abresch's solutions are very similar in structure to those above. In particular, they have a 'separated variable' form, being given in terms of single-variable functions $f(s), g(t)$ rather than two-variable functions, and f and g may be written explicitly using Jacobi elliptic functions.

We can also exploit the analogy in another way. Kapouleas [19] used analytic methods to construct examples of compact CMC surfaces Σ in $\mathbb{R}^3$ for any genus $g \geqslant 3$. It seems very likely that one could use Kapouleas' method to construct examples of higher genus (immersed) minimal Lagrangian surfaces in $\mathbb{CP}^2$, and minimal Legendrian surfaces in $\mathcal{S}^5$.

Kapouleas makes his examples by gluing together long segments of *Delaunay surfaces*, which are SO(2)-invariant CMC surfaces resembling a string of round 2-spheres joined by narrow, catenoid-like 'necks'. The appropriate analogues of Delaunay surfaces in our problem are Legendrian surfaces in $\mathcal{S}^5$ invariant under the U(1)-action $(z_1, z_2, z_3) \mapsto (e^{is} z_1, e^{-is} z_2, z_3)$, for $s \in \mathbb{R}$.

In the notation of §5.1–§5.3, these have $\theta = 0$ and $B = -1$. When the remaining parameter $C \in [-1, 1]$ is nonzero and small, the corresponding minimal Legendrian surfaces resemble chains of round Legendrian $\mathcal{S}^2$'s in $\mathcal{S}^5$ joined by small necks.

6 Interpretation using integrable systems

In Theorem 5.4 we constructed families of conformal harmonic maps $\phi : \mathbb{R}^2 \to \mathcal{S}^5$ and $\psi : \mathbb{R}^2 \to \mathbb{CP}^2$. We shall now analyze these in the integrable systems framework described in §3 and §4. We will show that they are generically superconformal, and explicitly determine their harmonic sequences, Toda and Tzitzéica solutions, loops of flat connections, polynomial Killing fields, and spectral curves. This goes some way towards redressing the 'dearth of examples' of

superconformal harmonic tori referred to by Bolton and Woodward [10, p. 76]. We shall use the notation of §5.1–§5.3 throughout.

6.1 The harmonic sequence of ψ

In the situation of §3.1, take U to be $\mathbb{R}^2$ with complex coordinate $z = s + it$. Then $\frac{\partial}{\partial z} = \frac{1}{2}\frac{\partial}{\partial s} - \frac{i}{2}\frac{\partial}{\partial t}$ and $\frac{\partial}{\partial \bar z} = \frac{1}{2}\frac{\partial}{\partial s} + \frac{i}{2}\frac{\partial}{\partial t}$. Thus by (24), (25), (28) and the definition $\phi(s,t) = \Phi(1,s,t)$ we have

$$\begin{aligned}\tfrac{\partial\phi}{\partial z} &= \tfrac{1}{2\sqrt{3}}(\beta_1\overline{y_2y_3}z_1 - i\gamma_1 y_1\overline{z_2z_3}, \beta_2\overline{y_3y_1}z_2 - i\gamma_2y_2\overline{z_3z_1}, \beta_3\overline{y_1y_2}z_3 - i\gamma_3y_3\overline{z_1z_2}),\\ \tfrac{\partial\phi}{\partial \bar z} &= \tfrac{1}{2\sqrt{3}}(\beta_1\overline{y_2y_3}z_1 + i\gamma_1 y_1\overline{z_2z_3}, \beta_2\overline{y_3y_1}z_2 + i\gamma_2y_2\overline{z_3z_1}, \beta_3\overline{y_1y_2}z_3 + i\gamma_3y_3\overline{z_1z_2}).\end{aligned}$$

Calculation using (26) and (27) shows that $\langle \frac{\partial\phi}{\partial z}, \phi\rangle = \langle \frac{\partial\phi}{\partial \bar z}, \phi\rangle = 0$. Also, using (40) we find that $\left|\frac{\partial\phi}{\partial z}\right|^2 = \left|\frac{\partial\phi}{\partial \bar z}\right|^2 = a + bv(s) + cw(t)$.

As $\langle \frac{\partial\phi}{\partial \bar z}, \phi\rangle = 0$, by definition ϕ is a holomorphic section of the holomorphic line bundle L_0 over $\mathbb{C}$ associated to $\psi_0 = \psi : \mathbb{C} \to \mathbb{CP}^2$. Therefore, from §3.1, there exists a unique sequence of maps $\phi_k : \mathbb{C} \to \mathbb{C}^3$ with $\phi_0 = \phi$, which satisfy (2), and the harmonic sequence (ψ_k) of ψ is given by $\psi_k = [\phi_k]$.

From (2) we see that $\phi_{-1} = -|\phi_0|^2 \left|\frac{\partial\phi_0}{\partial \bar z}\right|^{-2}\frac{\partial\phi_0}{\partial \bar z}$ and $\phi_1 = \frac{\partial\phi_0}{\partial z}$, since $|\phi_0| \equiv 1$. Thus the equations above give

$$\begin{aligned}\phi_{-1} &= -\frac{1}{2\sqrt{3}(a+bv+cw)}(\beta_1\overline{y_2y_3}z_1 + i\gamma_1y_1\overline{z_2z_3}, \beta_2\overline{y_3y_1}z_2 + i\gamma_2y_2\overline{z_3z_1},\\ &\qquad \beta_3\overline{y_1y_2}z_3 + i\gamma_3y_3\overline{z_1z_2}),\end{aligned} \tag{45}$$

$$\phi_0 = \tfrac{1}{\sqrt{3}}\big(y_1z_1, y_2z_2, y_3z_3\big), \tag{46}$$

$$\begin{aligned}\phi_1 &= \tfrac{1}{2\sqrt{3}}(\beta_1\overline{y_2y_3}z_1 - i\gamma_1y_1\overline{z_2z_3}, \beta_2\overline{y_3y_1}z_2 - i\gamma_2y_2\overline{z_3z_1},\\ &\qquad \beta_3\overline{y_1y_2}z_3 - i\gamma_3y_3\overline{z_1z_2}).\end{aligned} \tag{47}$$

These satisfy

$$|\phi_{-1}|^2 = (a+bv+cw)^{-1}, \quad |\phi_0|^2 = 1 \quad \text{and} \quad |\phi_1|^2 = a+bv+cw. \tag{48}$$

From (2) and the equation $|\phi_1|^2 = a + bv + cw$ we see that

$$\phi_2 = \frac{\partial\phi_1}{\partial z} - \frac{\partial}{\partial z}\big(\log(a+bv+cw)\big)\phi_1.$$

Substituting in for ϕ_1 from (47) gives a long and complicated expression for ϕ_2. After much calculation using equations (24)–(27), (35)–(36), (41)–(43) and other identities satisfied by β_j, γ_j and a, b, c, one can prove that

$$\phi_2 = \xi\phi_{-1}, \quad \text{where} \quad \xi = cC + ibB. \tag{49}$$

We can now identify the harmonic sequence of ψ.

Proposition 6.1 *If bB and cC are not both zero then $\psi : \mathbb{R}^2 \to \mathbb{CP}^2$ is superconformal, and has harmonic sequence (ψ_k) given by*

$$\begin{aligned}\psi_{3k-1}(s,t) = \big[&\beta_1\overline{y_2y_3}z_1 + i\gamma_1 y_1\overline{z_2z_3}, \beta_2\overline{y_3y_1}z_2 + i\gamma_2 y_2\overline{z_3z_1}, \\ &\beta_3\overline{y_1y_2}z_3 + i\gamma_3 y_3\overline{z_1z_2}\big],\end{aligned} \tag{50}$$

$$\psi_{3k}(s,t) = \big[y_1z_1, y_2z_2, y_3z_3\big], \tag{51}$$

$$\begin{aligned}\psi_{3k+1}(s,t) = \big[&\beta_1\overline{y_2y_3}z_1 - i\gamma_1 y_1\overline{z_2z_3}, \beta_2\overline{y_3y_1}z_2 - i\gamma_2 y_2\overline{z_3z_1}, \\ &\beta_3\overline{y_1y_2}z_3 - i\gamma_3 y_3\overline{z_1z_2}\big],\end{aligned} \tag{52}$$

for all $k \in \mathbb{Z}$. If $bB = cC = 0$ then ψ is isotropic, with finite harmonic sequence $\psi_{-1}, \psi_0, \psi_1$ given by equations (50)–(52) *with $k = 0$.*

Proof. Since $\phi_2 = \xi\phi_{-1}$ where $\xi = cC + ibB$ by (49), if $\xi \neq 0$ then the sequence (ϕ_k) exists for all k and is given by

$$\phi_{3k-1} = \xi^k\phi_{-1}, \quad \phi_{3k} = \xi^k\phi_0 \quad \text{and} \quad \phi_{3k+1} = \xi^k\phi_1.$$

Since $\psi_k = [\phi_k]$, equations (50)–(52) follow from (45)–(47). Thus ψ is non-isotropic, as ψ_k exists for all k. But any conformal map $\psi : S \to \mathbb{CP}^2$ is isotropic or superconformal from §3.1, so ψ is superconformal.

If on the other hand $\xi = 0$ then $\phi_2 = 0$, so ψ_2 does not exist. Thus ψ is isotropic. By (45)–(47), ψ_{-1}, ψ_0 and ψ_1 exist and are given by equations (50)–(52) with $k = 0$. But the harmonic sequence of an isotropic map $\psi : S \to \mathbb{CP}^m$ has length at most $m + 1$, so this is the whole of the harmonic sequence. $\square$

In the case when $\xi = 0$ and ψ is isotropic, ψ_{-1} is holomorphic and ψ_1 antiholomorphic. This is not obvious, but may be proved directly. For instance, when $B = C = 0$ we may take the y_j and z_j to be *real*. Then ψ maps to $\mathbb{RP}^2$ in $\mathbb{CP}^2$, and both ψ_1 and ψ_{-1} map to the conic $\big\{[w_0, w_1, w_2] \in \mathbb{CP}^2 : w_0^2 + w_1^2 + w_2^2 = 0\big\}$, with $\psi_{-1} = \overline{\psi}_1$.

6.2 Solutions of the Toda lattice and Tzitzéica equations

In the rest of the section we assume that $\xi = cC + ibB \neq 0$, so that ψ is superconformal. Following §3.2, we shall construct a solution of the Toda lattice equations for SU(3) out of ψ. The first thing to do is to find a *special* holomorphic coordinate z' on $\mathbb{C}$, that is, one in which $\xi' = 1$ and the ϕ'_k are periodic with period 3. By (5), $z' = z'(z)$ is special if

$$\xi' = \big(\tfrac{\partial z'}{\partial z}\big)^{-3}\xi = 1.$$

Thus we need $\frac{\partial z'}{\partial z} = \xi^{1/3}$ for some fixed complex cube root $\xi^{1/3}$ of ξ. So define $z' = \xi^{1/3}(s + it)$. Then z' is a special holomorphic coordinate on $\mathbb{C}$.

Working with respect to z' rather than z, we get a new sequence (ϕ'_k) rather than (ϕ_k), with $\phi'_k = -i\xi^{-k/3}\phi_k$. Thus from (45)–(48) we get

$$\begin{aligned}\phi'_{3k-1} &= \frac{i\xi^{1/3}}{2\sqrt{3}(a+bv+cw)}\big(\beta_1\overline{y_2y_3}z_1+i\gamma_1y_1\overline{z_2z_3},\beta_2\overline{y_3y_1}z_2+i\gamma_2y_2\overline{z_3z_1}, \\ &\qquad \beta_3\overline{y_1y_2}z_3+i\gamma_3y_3\overline{z_1z_2}\big),\end{aligned} \tag{53}$$

$$\phi'_{3k} = \frac{-i}{\sqrt{3}}\big(y_1z_1,y_2z_2,y_3z_3\big), \tag{54}$$

$$\begin{aligned}\phi'_{3k+1} &= \frac{-i\xi^{-1/3}}{2\sqrt{3}}\big(\beta_1\overline{y_2y_3}z_1-i\gamma_1y_1\overline{z_2z_3},\beta_2\overline{y_3y_1}z_2-i\gamma_2y_2\overline{z_3z_1}, \\ &\qquad \beta_3\overline{y_1y_2}z_3-i\gamma_3y_3\overline{z_1z_2}\big),\end{aligned} \tag{55}$$

$$\begin{aligned}\text{with}\quad & |\phi'_{3k-1}|^2 = |\xi|^{2/3}(a+bv+cw)^{-1}, \quad |\phi'_{3k}|^2 = 1 \\ \text{and}\quad & |\phi'_{3k+1}|^2 = |\xi|^{-2/3}(a+bv+cw) \quad \text{for all } k\in\mathbb{Z}.\end{aligned} \tag{56}$$

Here we have multiplied by $-i$ because then $\det(\phi'_0\phi'_1\phi'_2) \equiv 1$, as in (6). Thus the ϕ'_k satisfy all the conditions on the ϕ_k in §3.1–§3.2. So from §3.2 if we define $\chi_k = |\phi'_k|^2$, then the χ_k satisfy the Toda lattice equations for SU(3) with respect to z'. Therefore by (56) we have proved:

Proposition 6.2 *In the situation above, define $\chi_k : \mathbb{C} \to (0,\infty)$ by*

$$\begin{aligned}&\chi_{3k-1} = |\xi|^{2/3}(a+bv+cw)^{-1}, \quad \chi_{3k} = 1 \quad \textit{and} \\ &\chi_{3k+1} = |\xi|^{-2/3}(a+bv+cw) \quad \textit{for all } k\in\mathbb{Z}.\end{aligned} \tag{57}$$

Then the χ_k satisfy the Toda lattice equations for SU(3) *with respect to $z' = \xi^{1/3}(s+it)$. In terms of s,t, this means that $\chi_0\chi_1\chi_2 \equiv 1$, $\chi_{k+3} = \chi_k$ and*

$$\frac{1}{4|\xi|^{2/3}}\left(\frac{\partial^2}{\partial s^2}+\frac{\partial^2}{\partial t^2}\right)\big(\log\chi_k\big) = \chi_{k+1}\chi_k^{-1} - \chi_k\chi_{k-1}^{-1} \quad \textit{for all } k\in\mathbb{Z}. \tag{58}$$

Here (58) holds because $\frac{\partial^2}{\partial z'\partial\bar{z}'} = \frac{1}{4|\xi|^{2/3}}\big(\frac{\partial^2}{\partial s^2}+\frac{\partial^2}{\partial t^2}\big)$. One can verify (58) explicitly using equations (35)–(36), (41)–(43), (57) and various identities between the β_j,γ_j,B,C,a,b and c. The proposition defines a simple class of doubly-periodic solutions χ_k of the Toda lattice equations for SU(3). From §4.1 we deduce:

Corollary 6.3 *Define $f : \mathbb{C} \to (0,\infty)$ by $f = \log(a+bv+cw) - \frac{2}{3}\log|\xi|$. Then f satisfies the Tzitzéica equation* (22) *with respect to $z' = \xi^{1/3}(s+it)$.*

Note that the functions $v(s)$, $w(t)$ may be written in terms of Jacobi elliptic functions as in §5.2, and so the solutions in the last two results are entirely explicit. They have a 'separated variable' form, that is, they are written in terms of single-variable functions $v(s)$ and $w(t)$, rather than more general two-variable functions $u(s,t)$. The author is not sure whether these solutions are already known.

6.3 Loops of flat connections and polynomial Killing fields

For the rest of §6 we will work with the special coordinate $z = \xi^{1/3}(s+it)$, dropping the notation z'. From §3.3, the *Toda frame* $F : \mathbb{R}^2 \to \mathrm{SU}(3)$ of ψ is given by $F = (f_0 f_1 f_2)$, where $f_k = |\phi'_k|^{-1}\phi'_k$. Using equations (53)–(56) we may write F down explicitly, but we will not do so as the expression is complicated. Then $\alpha = F^{-1}\mathrm{d}F$ is a flat SU(3) connection matrix on $\mathbb{R}^2$.

As in §3.4, we may extend $\mathrm{d}+\alpha$ to a loop of flat SU(3)-connections $\mathrm{d}+\alpha_\lambda$ for $\lambda \in \mathbb{C}$ with $|\lambda| = 1$. We shall write α_λ out explicitly. Decompose α_λ as

$$\alpha_\lambda = (\alpha'_1\lambda + \alpha'_0)\mathrm{d}z + (\alpha''_{-1}\lambda^{-1} + \alpha''_0)\mathrm{d}\bar{z}, \tag{59}$$

as in (11). Then from (9) and (57) we find that

$$\alpha'_1 = r^{-1/3}\begin{pmatrix} 0 & 0 & f^{1/2} \\ f^{1/2} & 0 & 0 \\ 0 & rf^{-1} & 0 \end{pmatrix}, \qquad \alpha'_0 = \tfrac{1}{2}\begin{pmatrix} 0 & 0 & 0 \\ 0 & \frac{\partial}{\partial z}(\log f) & 0 \\ 0 & 0 & -\frac{\partial}{\partial z}(\log f) \end{pmatrix}, \tag{60}$$

$$\alpha''_{-1} = -r^{-1/3}\begin{pmatrix} 0 & f^{1/2} & 0 \\ 0 & 0 & rf^{-1} \\ f^{1/2} & 0 & 0 \end{pmatrix}, \qquad \alpha''_0 = \tfrac{1}{2}\begin{pmatrix} 0 & 0 & 0 \\ 0 & -\frac{\partial}{\partial \bar{z}}(\log f) & 0 \\ 0 & 0 & \frac{\partial}{\partial \bar{z}}(\log f) \end{pmatrix}, \tag{61}$$

where $f = a + bv + cw$ and $r = |\xi|$.

We shall now construct a *polynomial Killing field* τ for ψ, as in §3.5, which is in fact the nontrivial polynomial Killing field of lowest degree.

Theorem 6.4 *Write $\xi = r\mathrm{e}^{i\theta}$ for $r > 0$ and $\theta \in \mathbb{R}$. Define functions $f, h : \mathbb{R}^2 \to \mathbb{R}$ and $g : \mathbb{R}^2 \to \mathbb{C}$ by*

$$f = a + bv + cw, \quad g = \frac{1}{2f^{1/2}}\Big(-b\frac{\mathrm{d}v}{\mathrm{d}s} + ic\frac{\mathrm{d}w}{\mathrm{d}t}\Big), \quad h = \frac{1}{12f}\Big(-b\frac{\mathrm{d}^2v}{\mathrm{d}s^2} + c\frac{\mathrm{d}^2w}{\mathrm{d}t^2}\Big), \tag{62}$$

and let $\tau = \sum_{n=-2}^{2}\lambda^n\tau_n$, where

$$\tau_2 = i\mathrm{e}^{2i\theta/3}\begin{pmatrix} 0 & rf^{-1/2} & 0 \\ 0 & 0 & f \\ rf^{-1/2} & 0 & 0 \end{pmatrix}, \qquad \tau_1 = i\mathrm{e}^{i\theta/3}\begin{pmatrix} 0 & 0 & g \\ -g & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \tag{63}$$

$$\tau_0 = i\begin{pmatrix} 2h & 0 & 0 \\ 0 & -h & 0 \\ 0 & 0 & -h \end{pmatrix} \qquad \tau_{-1} = i\mathrm{e}^{-i\theta/3}\begin{pmatrix} 0 & -\bar{g} & 0 \\ 0 & 0 & 0 \\ \bar{g} & 0 & 0 \end{pmatrix}, \tag{64}$$

$$\textit{and} \qquad \tau_{-2} = i\mathrm{e}^{-2i\theta/3}\begin{pmatrix} 0 & 0 & rf^{-1/2} \\ rf^{-1/2} & 0 & 0 \\ 0 & f & 0 \end{pmatrix}. \tag{65}$$

Then τ is a real polynomial Killing field.

To prove the theorem one must show that the τ_n satisfy (16) and (17). This is a long but straightforward calculation, using equations (35), (36),

$$\frac{\partial}{\partial z} = \frac{1}{2r^{1/3}\mathrm{e}^{i\theta/3}}\Big(\frac{\partial}{\partial s} - i\frac{\partial}{\partial t}\Big) \quad \text{and} \quad \frac{\partial}{\partial \bar{z}} = \frac{1}{2r^{1/3}\mathrm{e}^{-i\theta/3}}\Big(\frac{\partial}{\partial s} + i\frac{\partial}{\partial t}\Big),$$

and identities satisfied by the β_j, γ_j, B, C and ξ, and we leave it to the reader.

Both α_λ and τ have an extra $\mathbb{Z}_2$-symmetry, which follows from the fact that $\chi_0 \equiv 1$. Define $\kappa : \mathfrak{gl}(3,\mathbb{C}) \to \mathfrak{gl}(3,\mathbb{C})$ by

$$\kappa : \begin{pmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{pmatrix} \mapsto - \begin{pmatrix} A_{11} & A_{31} & A_{21} \\ A_{13} & A_{33} & A_{23} \\ A_{12} & A_{32} & A_{22} \end{pmatrix}. \tag{66}$$

Then κ is a Lie algebra automorphism, and $\kappa^2 = 1$. It is easy to show from (60)–(61) and (63)–(65) that

$$\kappa(\alpha_\lambda) = \alpha_{-\lambda} \quad \text{and} \quad \kappa(\tau(\lambda)) = -\tau(-\lambda) \quad \text{for all } \lambda \in \mathbb{C}^*. \tag{67}$$

The action of κ on the algebra of polynomial Killing fields will induce the holomorphic involution ρ on the spectral curve discussed in §4.2.

We can now determine the algebra of polynomial Killing fields $\mathcal{A}$.

Theorem 6.5 *In the situation above, the algebra of polynomial Killing fields $\mathcal{A}$ is generated by τ, $\lambda^3 I$ and $\lambda^{-3} I$.*

Proof. Let $\mathcal{A}'$ be the subalgebra of $\mathcal{A}$ generated by τ, $\lambda^3 I$ and $\lambda^{-3} I$, and suppose for a contradiction that $\mathcal{A}' \neq \mathcal{A}$. Let $\eta \in \mathcal{A} \setminus \mathcal{A}'$, and take η to be real, and of lowest degree d. That is, $\eta = \sum_{n=-d}^{d} \lambda^n \eta_n$ with $\eta_{-n} = -\bar{\eta}_n^T$ for $n = 0, \ldots, d$, and every polynomial Killing field of degree less than d lies in $\mathcal{A}'$.

As $\eta_{d+1} = 0$, equations (16) with $n = d+1$ and (17) with $n = d$ show that η_d satisfies

$$[\eta_d, \alpha_1'] = 0 \quad \text{and} \quad \frac{\partial \eta_d}{\partial \bar{z}} = [\eta_d, \alpha_0''\,]. \tag{68}$$

Divide into the three cases (a) $d = 3k$, (b) $d = 3k+1$, and (c) $d = 3k+2$ for some $k = 0, 1, 2, \ldots$. We will prove a contradiction in each case in turn.

In case (a), equation (13) implies that η_d is diagonal, and then as f is nonzero, the first equations of (60) and (68) show that η_d is a multiple of the identity. So write $\eta_d = \epsilon I$ for some $\epsilon : \mathbb{R}^2 \to \mathbb{C}$. Taking the trace of equations (16) and (17) for $n = d$ gives $\frac{\partial \epsilon}{\partial z} = \frac{\partial \epsilon}{\partial \bar{z}} = 0$, as the trace of any commutator is zero. Thus ϵ is constant, and $\eta_d = \epsilon I$, $\eta_{-d} = -\bar{\epsilon} I$.

For $k > 0$, consider $\eta' = \eta - \epsilon(\lambda^3 I)^k + \bar{\epsilon}(\lambda^{-3} I)^{-k}$. This is a polynomial Killing field of degree less than d, as we have cancelled the terms in $\lambda^{\pm d}$. Therefore $\eta' \in \mathcal{A}'$. But $\eta = \eta' + \epsilon(\lambda^3 I)^k - \bar{\epsilon}(\lambda^{-3} I)^{-k}$, so $\eta \in \mathcal{A}'$, a contradiction. Also, when $k = 0$ we have $\eta = \epsilon I \in \mathcal{A}'$. This eliminates case (a).

Similarly, in case (b), equation (13) and the first equations of (60) and (68) imply that

$$\eta_d = \epsilon r^{-1/3} \begin{pmatrix} 0 & 0 & f^{1/2} \\ f^{1/2} & 0 & 0 \\ 0 & r f^{-1} & 0 \end{pmatrix},$$

for some function $\epsilon : \mathbb{R}^2 \to \mathbb{C}$. The second equation of (56) is equivalent to $\frac{\partial \epsilon}{\partial \bar{z}} = 0$, so that ϵ is holomorphic. Using the fact that $F_\lambda \eta F_\lambda^{-1}$ is independent of z one can show that ϵ must be constant. This determines η_d and η_{-d}.

By (63), the leading term of τ^2 is

$$-\lambda^4 e^{4i\theta/3} \begin{pmatrix} 0 & 0 & rf^{1/2} \\ rf^{1/2} & 0 & 0 \\ 0 & r^2 f^{-1} & 0 \end{pmatrix}.$$

Suppose for the moment that $d \geqslant 7$, so that $k \geqslant 2$. Consider

$$\eta' = \eta + (\lambda^3 I)^{k-1} \xi^{-4/3} \epsilon \tau^2 - (\lambda^{-3} I)^{k-1} \bar{\xi}^{-4/3} \bar{\epsilon} \tau^2.$$

We have cancelled the terms in $\lambda^{\pm d}$, so η' is a polynomial Killing field of degree less than d, and lies in $\mathcal{A}'$. So η lies in $\mathcal{A}'$, a contradiction.

The cases $d = 1$ and $d = 4$ must be dealt with separately. By explicit calculation we prove that η is a multiple of I when $d = 1$, and a linear combination of $I, \lambda^{\pm 3} I$, τ and τ^2 when $d = 4$. So $\eta \in \mathcal{A}'$, finishing case (b).

In the same way, in case (c) we find that

$$\eta_d = \epsilon r^{-2/3} \begin{pmatrix} 0 & rf^{-1/2} & 0 \\ 0 & 0 & f \\ rf^{-1/2} & 0 & 0 \end{pmatrix}$$

for some constant $\epsilon \in \mathbb{C}$. When $d \geqslant 5$ we define

$$\eta' = \eta + (\lambda^3 I)^k i \xi^{-2/3} \epsilon \tau + (\lambda^{-3} I)^k i \bar{\xi}^{-2/3} \bar{\epsilon} \tau,$$

and deduce that $\eta' \in \mathcal{A}'$, so that $\eta \in \mathcal{A}'$. The case $d = 2$ we deal with separately, by showing that η is a linear combination of τ and I, and so lies in $\mathcal{A}'$. This completes the proof. $\square$

We can use similar ideas to show that ψ is of *finite type*, as in §3.5. Define

$$\eta = (\xi^{-4/3} \lambda^3 - \bar{\xi}^{-4/3} \lambda^{-3}) \tau^2. \tag{69}$$

Then η is a real polynomial Killing field of degree 7, and (60) and (63) imply that $\eta_7 = \alpha'_1$ and $\eta_6 = 2\alpha'_0$. So, by definition, ψ is of finite type.

Furthermore, the proof of the theorem actually implies that *every* polynomial Killing field is of the form $P_0 I + P_1 \tau + P_2 \tau^2$, where P_0, P_1, P_2 are Laurent polynomials in $\lambda^{\pm 3}$. Writing τ^3 in this way, and using the $\mathbb{Z}_2$-symmetry (67) to eliminate some of the terms, we find that τ must satisfy a cubic equation

$$\tau^3 + D\tau + i(\xi^2 \lambda^6 + E + \bar{\xi}^2 \lambda^{-6}) I = 0 \tag{70}$$

for some $D, E \in \mathbb{R}$. Then $\mathcal{A}$ is the quotient of the free commutative algebra generated by $\lambda^{\pm 3} I$ and τ by the ideal generated by this equation.

6.4 The spectral curve

Now we can calculate the spectral curve of ψ, as in §3.6. Define

$$Y' = \big\{(\lambda,\mu)\in\mathbb{C}^*\times\mathbb{C} : \det\big(\mu I-\tau(\lambda,z)\big)=0\big\},$$

as in (20). Since $\mathcal{A}$ is generated by τ and $\lambda^{\pm 3}I$, this is biholomorphic to the curve Y of (18), and so the spectral curve $\tilde{Y}$ as defined by Ferus et al. [9, §5] is the compactification $\tilde{Y}$ of Y'.

Calculating using (63)–(65), we find that

$$\det(\mu I-\tau) = \mu^3 + D\mu + iE + i\xi^2\lambda^6 + i\bar{\xi}^2\lambda^{-6}, \quad \text{where} \tag{71}$$

$$D = f^2 + 2r^2 f^{-1} + 2|g|^2 + 3h^2 \quad \text{and} \tag{72}$$

$$E = -fg^2 - f\bar{g}^2 - 2f^2h + 2r^2f^{-1}h + 2|g|^2h + 2h^3. \tag{73}$$

As Y' is independent of $z\in\mathbb{C}$, the functions D,E are constant, which may be verified directly using (35)–(36), (62) and identities satisfied by the $\beta_j,\gamma_j,a,b,$ c,B,C and r.

We can find explicit expressions for these constants by putting $v=w=0$, which by (35)–(36) gives

$$\left(\tfrac{\mathrm{d}v}{\mathrm{d}s}\right)^2 = 4(1-B^2), \quad \left(\tfrac{\mathrm{d}w}{\mathrm{d}t}\right)^2 = 4(1-C^2) \quad \text{and} \quad \tfrac{\mathrm{d}^2v}{\mathrm{d}s^2} = \tfrac{\mathrm{d}^2w}{\mathrm{d}t^2} = 0.$$

Equation (62) gives values for f,g and h, and substituting these into (72) and (73) yields

$$D = a^2 + 2a^{-1}(b^2+c^2) \quad \text{and} \quad E = 2\big(b^2(1-B^2) - c^2(1-C^2)\big). \tag{74}$$

This proves that the spectral curve as defined by Ferus et al. [9, §5] is the compactification $\tilde{Y}$ of

$$Y' = \big\{(\lambda,\mu)\in\mathbb{C}^*\times\mathbb{C} : \mu^3 + D\mu + iE + i\xi^2\lambda^6 + i\bar{\xi}^2\lambda^{-6} = 0\big\}, \tag{75}$$

where D and E are given by (74). It can be shown using elementary algebraic geometry that $\tilde{Y}$ is nonsingular for generic D,E, with genus 10. Note that the equation satisfied by μ in (75) is the same as that satisfied by τ in (70).

However, McIntosh [22, 23, 24] uses a different definition of the spectral curve. To find it we replace λ^3 by λ in (75), giving

$$X' = \big\{(\lambda,\mu)\in\mathbb{C}^*\times\mathbb{C} : \mu^3 + D\mu + iE + i\xi^2\lambda^2 + i\bar{\xi}^2\lambda^{-2} = 0\big\}, \tag{76}$$

and McIntosh's spectral curve is the compactification $\tilde{X}$ of X'. For generic D,E it is nonsingular with genus 4. The involutions $\sigma:\tilde{X}\to\tilde{X}$ and $\rho:\tilde{X}\to\tilde{X}$ discussed in §3.6 and §4.2 act by

$$\rho:(\lambda,\mu)\mapsto(-\lambda,\mu) \quad \text{and} \quad \sigma:(\lambda,\mu)\mapsto(\bar{\lambda}^{-1},-\bar{\mu}). \tag{77}$$

It would be interesting to understand what properties of the spectral curve $\tilde{X}$ correspond to the fact that ψ is written in terms of single-variable functions $y_k(s)$ and $z_k(t)$, rather than more general two-variable functions of (s,t). Ian McIntosh has an explanation of this, which may appear elsewhere.

6.5 Interpretation using the ideas of §4

Finally we relate the calculations above to the material of §4. From §6.4 the spectral curve X as defined by McIntosh has genus 4. Thus in §4.3 we have $p = 4$ and $d = 2$. The parameter counts there show that the moduli space of all finite type genus 4 solutions of the Tzitzéica equation, up to translations in $\mathbb{R}^2$, should have dimension 4. All of them are expected to be doubly-periodic. For the corresponding maps $\phi : \mathbb{R}^2 \to \mathcal{S}^5$ and $\psi : \mathbb{R}^2 \to \mathbb{CP}^2$ to be doubly-periodic is 4 rationality conditions.

Now the family of genus 4 solutions of the Tzitzéica equations constructed in Corollary 6.3 depends up to translations in $\mathbb{R}^2$ on the 3 parameters θ, B, C of §5. Thus, we have not constructed all the genus 4 Tzitzéica solutions, but only a codimension 1 subset of them. This agrees with the analysis of §5.4, where we were unable to solve the double-periodicity conditions in general, because they amounted to 4 rationality conditions on 3 variables.

Here are two ways of thinking about why the construction yields only a codimension 1 subset of the Tzitzéica solutions. Firstly, our solutions have a 'separated variable' form, being written in terms of functions $v(s), w(t)$. It follows that the period vectors of the doubly-periodic Tzitzéica solutions will point along the s and t axes, and so be perpendicular in $\mathbb{R}^2$. However, the general genus 4 Tzitzéica solution will have period vectors which are not orthogonal, and to require them to be orthogonal is a codimension 1 condition.

Secondly, although the moduli space of quadruples $(\tilde{X}, \rho, \sigma, \pi)$ with $\tilde{X}$ genus 4 is four-dimensional, the subset which can be defined by an equation of the form (76) is only 3-dimensional. In §6.3 we saw that our solutions admit a degree 2 polynomial Killing field τ, which satisfies a cubic equation over $\mathbb{C}[\lambda^3 I, \lambda^{-3} I]$. It is this cubic equation which gives X' the simple form (76).

So we conclude that although the family of genus 4 Tzitzéica solutions has dimension 4, only a 3-dimensional subfamily of these admit a degree 2 polynomial Killing field τ, and it is this which is responsible for the special form (76) of the spectral curve, and for the other nice behaviour of these examples. For generic genus 4 Tzitzéica solutions the first non-trivial polynomial Killing field will be of higher degree, and so the spectral curve will be given by a (singular) equation of higher-degree in $\lambda^{\pm 2}$.

7 Extension to three variables

Next we generalize Theorem 5.1 to a construction of special Lagrangian 3-folds in $\mathbb{C}^3$ in which all three variables r, s, t enter in a nontrivial way. The proof is similar to that of Theorem 5.1, so we will be brief.

Theorem 7.1 *Let $\alpha_1, \alpha_2, \alpha_3$, $\beta_1, \beta_2, \beta_3$ and $\gamma_1, \gamma_2, \gamma_3$ be real numbers with not all α_j, not all β_j and not all γ_j zero, such that*

$$\begin{aligned} \alpha_1\beta_1 + \alpha_2\beta_2 + \alpha_3\beta_3 = 0, \qquad\qquad & \alpha_1\gamma_1 + \alpha_2\gamma_2 + \alpha_3\gamma_3 = 0, \\ \beta_1\gamma_1 + \beta_2\gamma_2 + \beta_3\gamma_3 = 0 \quad \text{and} \quad & \alpha_1\beta_1\gamma_1 + \alpha_2\beta_2\gamma_2 + \alpha_3\beta_3\gamma_3 = 0. \end{aligned} \tag{78}$$

Let I,J,K be open intervals in $\mathbb{R}$. Suppose that $x_1,x_2,x_3:I\rightarrow\mathbb{C}$ and $u:I\rightarrow\mathbb{R}$ are functions of r, that $y_1,y_2,y_3:J\rightarrow\mathbb{C}$ and $v:J\rightarrow\mathbb{R}$ are functions of s, and $z_1,z_2,z_3:K\rightarrow\mathbb{C}$ and $w:K\rightarrow\mathbb{R}$ functions of t, satisfying

$$\frac{\mathrm{d}x_1}{\mathrm{d}r}=\alpha_1\,\overline{x_2x_3},\qquad \frac{\mathrm{d}x_2}{\mathrm{d}r}=\alpha_2\,\overline{x_3x_1},\qquad \frac{\mathrm{d}x_3}{\mathrm{d}r}=\alpha_3\,\overline{x_1x_2},\tag{79}$$

$$\frac{\mathrm{d}y_1}{\mathrm{d}s}=\beta_1\,\overline{y_2y_3},\qquad \frac{\mathrm{d}y_2}{\mathrm{d}s}=\beta_2\,\overline{y_3y_1},\qquad \frac{\mathrm{d}y_3}{\mathrm{d}s}=\beta_3\,\overline{y_1y_2},\tag{80}$$

$$\frac{\mathrm{d}z_1}{\mathrm{d}t}=\gamma_1\,\overline{z_2z_3},\qquad \frac{\mathrm{d}z_2}{\mathrm{d}t}=\gamma_2\,\overline{z_3z_1},\qquad \frac{\mathrm{d}z_3}{\mathrm{d}t}=\gamma_3\,\overline{z_1z_2},\tag{81}$$

$$|x_1|^2=\alpha_1u+1,\qquad |x_2|^2=\alpha_2u+1,\qquad |x_3|^2=\alpha_3u+1,\tag{82}$$

$$|y_1|^2=\beta_1v+1,\qquad |y_2|^2=\beta_2v+1,\qquad |y_3|^2=\beta_3v+1,\tag{83}$$

$$|z_1|^2=\gamma_1w+1,\qquad |z_2|^2=\gamma_2w+1,\qquad |z_3|^2=\gamma_3w+1.\tag{84}$$

If (79)–(81) *hold for all r,s,t and* (82)–(84) *hold for some r,s,t, then* (82)–(84) *hold for all r,s,t, for some functions u,v,w. Define $\Phi:I\times J\times K\rightarrow\mathbb{C}^3$ by*

$$\Phi:(r,s,t)\mapsto\bigl(x_1(r)y_1(s)z_1(t),x_2(r)y_2(s)z_2(t),x_3(r)y_3(s)z_3(t)\bigr).\tag{85}$$

Define a subset N of $\mathbb{C}^3$ by

$$N=\bigl\{\Phi(r,s,t):r\in I,\ s\in J,\ t\in K\bigr\}.\tag{86}$$

Then N is a special Lagrangian 3*-fold in $\mathbb{C}^3$.*

Proof. The first part of the theorem, that if (79)–(81) hold for all r,s,t and (82)–(84) for some r,s,t, then (82)–(84) hold for all r,s,t, follows as in Theorem 5.1. For the second part, we must prove that N is special Lagrangian wherever Φ is an immersion. As in Theorem 5.1, this holds if and only if

$$\omega\Bigl(\frac{\partial\Phi}{\partial r},\frac{\partial\Phi}{\partial s}\Bigr)\equiv\omega\Bigl(\frac{\partial\Phi}{\partial r},\frac{\partial\Phi}{\partial t}\Bigr)\equiv\omega\Bigl(\frac{\partial\Phi}{\partial s},\frac{\partial\Phi}{\partial t}\Bigr)\equiv 0\tag{87}$$

$$\text{and}\qquad \operatorname{Im}\Omega\Bigl(\frac{\partial\Phi}{\partial r},\frac{\partial\Phi}{\partial s},\frac{\partial\Phi}{\partial t}\Bigr)\equiv 0.\tag{88}$$

Using equations (79)–(81) and (85) we find that

$$\frac{\partial\Phi}{\partial r}=\bigl(\alpha_1\,\overline{x_2x_3}y_1z_1,\alpha_2\,\overline{x_3x_1}y_2z_2,\alpha_3\,\overline{x_1x_2}y_3z_3\bigr),\tag{89}$$

$$\frac{\partial\Phi}{\partial s}=\bigl(\beta_1x_1\,\overline{y_2y_3}z_1,\beta_2x_2\,\overline{y_3y_1}z_2,\beta_3x_3\,\overline{y_1y_2}z_3\bigr),\tag{90}$$

$$\frac{\partial\Phi}{\partial t}=\bigl(\gamma_1x_1y_1\,\overline{z_2z_3},\gamma_2x_2y_2\,\overline{z_3z_1},\gamma_3x_3y_3\,\overline{z_1z_2}\bigr).\tag{91}$$

Equations (89) and (90) give

$$\begin{aligned}\omega\bigl(\tfrac{\partial\Phi}{\partial r},\tfrac{\partial\Phi}{\partial s}\bigr)&=\operatorname{Im}(\overline{x_1x_2x_3}y_1y_2y_3)\bigl(\alpha_1\beta_1|z_1|^2+\alpha_2\beta_2|z_2|^2+\alpha_3\beta_3|z_3|^2\bigr)\\&=\operatorname{Im}(\overline{x_1x_2x_3}y_1y_2y_3)\bigl(\alpha_1\beta_1(\gamma_1w+1)+\alpha_2\beta_2(\gamma_2w+1)+\alpha_3\beta_3(\gamma_3w+1)\bigr)\\&=\operatorname{Im}(\overline{x_1x_2x_3}y_1y_2y_3)\bigl(\alpha_1\beta_1+\alpha_2\beta_2+\alpha_3\beta_3+w(\alpha_1\beta_1\gamma_1+\alpha_2\beta_2\gamma_2+\alpha_3\beta_3\gamma_3)\bigr)=0,\end{aligned}$$

using (84) in the second line and (78) in the third. This proves the first equation of (87). The second and third follow in a similar way.

To prove (88), observe that

$$\Omega\Big(\frac{\partial\Phi}{\partial r},\frac{\partial\Phi}{\partial s},\frac{\partial\Phi}{\partial t}\Big)=\Big|\frac{\partial\Phi}{\partial r}\ \frac{\partial\Phi}{\partial s}\ \frac{\partial\Phi}{\partial t}\Big|=\begin{vmatrix}\alpha_1\,\overline{x_2x_3}y_1z_1 & \beta_1x_1\,\overline{y_2y_3}z_1 & \gamma_1x_1y_1\,\overline{z_2z_3}\\ \alpha_2\,\overline{x_3x_1}y_2z_2 & \beta_2x_2\,\overline{y_3y_1}z_2 & \gamma_2x_2y_2\,\overline{z_3z_1}\\ \alpha_3\,\overline{x_1x_2}y_3z_3 & \beta_3x_3\,\overline{y_1y_2}z_3 & \gamma_3x_3y_3\,\overline{z_1z_2}\end{vmatrix}$$

$$\begin{aligned}=\big(&\alpha_1|x_2x_3|^2\beta_2|y_3y_1|^2\gamma_3|z_1z_2|^2+\alpha_2|x_3x_1|^2\beta_3|y_1y_2|^2\gamma_1|z_2z_3|^2\\ &+\alpha_3|x_1x_2|^2\beta_1|y_2y_3|^2\gamma_2|z_3z_1|^2-\alpha_1|x_2x_3|^2\beta_3|y_1y_2|^2\gamma_2|z_3z_1|^2\\ &-\alpha_2|x_3x_1|^2\beta_1|y_2y_3|^2\gamma_3|z_1z_2|^2-\alpha_3|x_1x_2|^2\beta_2|y_3y_1|^2\gamma_1|z_2z_3|^2\big).\end{aligned}$$

Thus $\Omega\big(\frac{\partial\Phi}{\partial r},\frac{\partial\Phi}{\partial s},\frac{\partial\Phi}{\partial t}\big)$ is real, and so $\operatorname{Im}\Omega\big(\frac{\partial\Phi}{\partial r},\frac{\partial\Phi}{\partial s},\frac{\partial\Phi}{\partial t}\big)=0$. □

Here are a few comments on the theorem.

(a) In Theorem 5.1 we took the ranges of s,t to be $\mathbb{R}$, but here we take r,s,t in intervals I,J,K in $\mathbb{R}$. This is because, by an argument in [14, Prop. 7.11], the conditions $\beta_1+\beta_2+\beta_3=0$ and $\gamma_1+\gamma_2+\gamma_3=0$ imply that solutions of (24) and (25) in some open interval extend automatically to all of $\mathbb{R}$.

However, in Theorem 7.1 we do not assume that $\alpha_1+\alpha_2+\alpha_3=0$, and so it could happen that $\alpha_1,\alpha_2,\alpha_3$ all have the same sign. In this case, solutions x_j to (79) will in general exist in some open interval $I\subset\mathbb{R}$ with $|x_j|\to\infty$ at the endpoints of I, so that they do not extend to $\mathbb{R}$. The same applies to (80) and (81).

(b) As in §5.2 we can write the x_k,y_k and z_k entirely explicitly in terms of integrals involving the Jacobi elliptic functions.

(c) As in Theorem 5.4, in the situation of Theorem 7.1, $\frac{\partial\Phi}{\partial r}$, $\frac{\partial\Phi}{\partial s}$ and $\frac{\partial\Phi}{\partial t}$ are always complex orthogonal. But in general they are not of the same length, so Φ is not conformal.

(d) We may recover Theorem 5.1 from Theorem 7.1 as follows. Put $\alpha_1=\alpha_2=\alpha_3=1$, so that (78) becomes equivalent to (23). Define

$$I=(-\infty,0),\quad x_1(r)=x_2(r)=x_3(r)=-r^{-1}\quad\text{and}\quad u(r)=r^{-2}-1,$$

and $J=K=\mathbb{R}$. Then (79) and (82) hold, and Theorem 7.1 becomes equivalent to Theorem 5.1, but with a different parametrization for r.

7.1 Description of the family of SL 3-folds

We shall now describe the family of SL 3-folds resulting from Theorem 7.1. We begin by studying the set of solutions $\alpha_j,\beta_j,\gamma_j$ to (78). Define vectors

$$\begin{aligned}&\boldsymbol{\alpha}=(\alpha_1,\alpha_2,\alpha_3), &&\boldsymbol{\beta}=(\beta_1,\beta_2,\beta_3), &&\boldsymbol{\gamma}=(\gamma_1,\gamma_2,\gamma_3),\\ &\boldsymbol{\alpha\beta}=(\alpha_1\beta_1,\alpha_2\beta_2,\alpha_3\beta_3), &&\boldsymbol{\alpha\gamma}=(\alpha_1\gamma_1,\alpha_2\gamma_2,\alpha_3\gamma_3), &&\boldsymbol{\beta\gamma}=(\beta_1\gamma_1,\beta_2\gamma_2,\beta_3\gamma_3)\end{aligned}$$

in $\mathbb{R}^3$. Rescaling $\boldsymbol{\alpha}, \boldsymbol{\beta}$ and $\boldsymbol{\gamma}$ has no effect on the SL 3-folds constructed in Theorem 7.1, so let us assume $\boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\gamma}$ are unit vectors. We will show that a generic choice of $\boldsymbol{\alpha}$ determines $\boldsymbol{\beta}, \boldsymbol{\gamma}$, essentially uniquely.

Proposition 7.2 *Let $\boldsymbol{\alpha}$ be a unit vector in $\mathbb{R}^3$, with $\alpha_1, \alpha_2, \alpha_3$ distinct and nonzero. Then there exist unit vectors $\boldsymbol{\beta}, \boldsymbol{\gamma}$ satisfying* (78)*, which are unique up to sign and exchanging $\boldsymbol{\beta}, \boldsymbol{\gamma}$.*

Proof. Equation (78) implies that $\boldsymbol{\alpha}, \boldsymbol{\beta}$ and $\boldsymbol{\alpha}\boldsymbol{\beta}$ are orthogonal to $\boldsymbol{\gamma}$. As $\boldsymbol{\gamma} \neq 0$, it follows that $\boldsymbol{\alpha}, \boldsymbol{\beta}$ and $\boldsymbol{\alpha}\boldsymbol{\beta}$ are linearly dependent. Therefore $\det\bigl(\boldsymbol{\alpha}\ \boldsymbol{\beta}\ \boldsymbol{\alpha}\boldsymbol{\beta}\bigr) = 0$. This may be rewritten in matrix form as

$$Q(\boldsymbol{\beta}) = \tfrac{1}{2} \begin{pmatrix} \beta_1 \\ \beta_2 \\ \beta_3 \end{pmatrix}^T \begin{pmatrix} 0 & \alpha_3(\alpha_2 - \alpha_1) & \alpha_2(\alpha_1 - \alpha_3) \\ \alpha_3(\alpha_2 - \alpha_1) & 0 & \alpha_1(\alpha_3 - \alpha_2) \\ \alpha_2(\alpha_1 - \alpha_3) & \alpha_1(\alpha_3 - \alpha_2) & 0 \end{pmatrix} \begin{pmatrix} \beta_1 \\ \beta_2 \\ \beta_3 \end{pmatrix} = 0. \tag{92}$$

Similar equations hold between the α_j and γ_j, and between the β_j and γ_j. Now the 3×3 matrix appearing in (92) has trace zero and determinant $2\alpha_1\alpha_2\alpha_3(\alpha_1 - \alpha_3)(\alpha_2 - \alpha_1)(\alpha_3 - \alpha_2)$. As by assumption $\alpha_1, \alpha_2, \alpha_3$ are distinct and nonzero, this determinant is nonzero. Hence Q is a trace-free, nondegenerate quadratic form on $\mathbb{R}^3$.

Therefore, $\boldsymbol{\beta}$ must be a unit vector in the intersection of the plane $\boldsymbol{\alpha} \cdot \boldsymbol{\beta} = 0$ and the quadric cone (92) in $\mathbb{R}^3$. Let $\boldsymbol{\alpha}^\perp$ be the plane perpendicular to $\boldsymbol{\alpha}$, and consider the restriction $Q|_{\boldsymbol{\alpha}^\perp}$ of Q to $\boldsymbol{\alpha}^\perp$. As $\boldsymbol{\alpha}$ is a unit vector, we have

$$0 = \mathrm{Tr}(Q) = \mathrm{Tr}\bigl(Q|_{\boldsymbol{\alpha}^\perp}\bigr) + Q(\boldsymbol{\alpha}).$$

But $Q(\boldsymbol{\alpha}) = 0$ by (92), so $Q|_{\boldsymbol{\alpha}^\perp}$ is trace-free.

Thus, by the classification of quadratic forms on $\mathbb{R}^2$, there exists an orthonormal basis $\boldsymbol{\beta}, \boldsymbol{\gamma}$ for $\boldsymbol{\alpha}^\perp$ such that $Q(x\boldsymbol{\beta} + y\boldsymbol{\gamma}) = cxy$ for some c and all x, y in $\mathbb{R}$. If $c = 0$ then $Q|_{\boldsymbol{\alpha}^\perp} = 0$, so Q is degenerate, a contradiction. So $c \neq 0$, and therefore $\boldsymbol{\beta}, \boldsymbol{\gamma}$ are unique up to sign and order, with $Q(\boldsymbol{\beta}) = Q(\boldsymbol{\gamma}) = 0$.

As $\boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\gamma}$ are orthonormal they automatically satisfy the first three equations of (78). But by construction we have arranged that $\boldsymbol{\alpha}, \boldsymbol{\beta}$ and $\boldsymbol{\alpha}\boldsymbol{\beta}$ are linearly dependent, so $\boldsymbol{\alpha}\boldsymbol{\beta} = x\boldsymbol{\alpha} + y\boldsymbol{\beta}$ for $x, y \in \mathbb{R}$. The fourth equation of (78) then follows from the second and third. $\square$

The moral of the proposition is that a generic choice of $\boldsymbol{\alpha}$ determines $\boldsymbol{\beta}$ and $\boldsymbol{\gamma}$ up to obvious symmetries. However, for a nongeneric choice of $\boldsymbol{\alpha}$ there can be more freedom in $\boldsymbol{\beta}$ and $\boldsymbol{\gamma}$. For instance, if we put $\boldsymbol{\alpha} = 3^{-1/2}(1,1,1)$ then $Q \equiv 0$, and $\boldsymbol{\beta}, \boldsymbol{\gamma}$ can be arbitrary orthonormal vectors in $\boldsymbol{\alpha}^\perp$.

We can now do a parameter count for the family of SL 3-folds coming from Theorem 7.1. The proposition shows that up to symmetries, the data $\alpha_j, \beta_j, \gamma_j$ has two interesting degrees of freedom. Also, as in Propositions 5.2 and 5.3 there exist constants $A, B, C \in \mathbb{R}$ such that

$$\mathrm{Im}(x_1x_2x_3) \equiv A, \quad \mathrm{Im}(y_1y_2y_3) \equiv B \quad \text{and} \quad \mathrm{Im}(z_1z_2z_3) \equiv C.$$

Together the $\alpha_j, \beta_j, \gamma_j$ and A, B, C determine N up to automorphisms of $\mathbb{C}^3$. Thus the construction of Theorem 7.1 yields a 5-dimensional family of SL 3-folds, up to automorphisms of $\mathbb{C}^3$.

We can also discuss the possible *signs* of the $\alpha_k, \beta_k, \gamma_k$. Suppose for simplicity that $\alpha_k, \beta_k, \gamma_k$ are all nonzero. Then the four equations of (78) constrain the signs of $\alpha_k, \beta_k, \gamma_k$, as in each equation the three terms cannot have the same sign, since their sum is zero. Now permuting $\boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\gamma}$, and reversing any of their signs, does not change the set of SL 3-folds constructed in Theorem 7.1.

Considering the constraints on the signs of the $\alpha_k, \beta_k, \gamma_k$, it is not difficult to show that by permuting and changing signs of $\boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\gamma}$ we may can arrange that the α_k are all positive, two of the β_k are positive and one negative, and two of the γ_k positive and one negative.

With this choice of signs, the argument in [14, Prop. 7.11] shows that solutions y_k, z_k to (80)–(81) and (83)–(84) automatically extend to $\mathbb{R}$, so we may take $J = K = \mathbb{R}$. However, solutions x_k to (79) and (82) generally exist only on a proper subinterval I of $\mathbb{R}$. Let us take I to be as large as possible.

The discussion of §5.4 suggests that we should try to arrange that the y_k are periodic in s and the z_k periodic in t. When this happens, ψ pushes down to an immersion $I \times T^2 \to \mathbb{C}^3$, whose image is a *closed* SL 3-fold in $\mathbb{C}^3$. The double-periodicity conditions in s, t in this case turn out to be equivalent to those in §5.4, and there are analogues of parts (a)–(c) of §5.4 in which one can prove they are soluble, which yield countably many families of closed, immersed SL 3-folds in $\mathbb{C}^3$ diffeomorphic to $T^2 \times \mathbb{R}$.

7.2 Conclusion: an open problem

Theorems 5.1 and 7.1 are clearly very similar. But in §6 we saw that the special Lagrangian cones of Theorem 5.1 can be put into a much larger integrable systems framework. Is there also an 'integrable systems' explanation for the SL 3-folds of Theorem 7.1? Certainly the solutions of Theorem 7.1 have many of the hallmarks of integrable systems: commuting o.d.e.s, elliptic functions, conserved quantities.

More generally, I suspect that in some sense, SL m-folds in $\mathbb{C}^m$ for $m \geqslant 3$ may constitute some kind of higher-dimensional integrable system.

The evidence for this is that there exist many interesting families of SL m-folds in $\mathbb{C}^m$ which can be written down explicitly, or have some other nice properties. For examples, see papers by the author [14, 15, 16, 17, 18], and others such as Harvey and Lawson [12, III.3], Haskins [13] and Bryant [5]. Also, when the special Lagrangian equations are reduced to an o.d.e., it often turns out to be a completely integrable Hamiltonian system, as in [14, §7.6].

I have no real idea of how to prove that the special Lagrangian equations are integrable, or even of exactly what it would mean for a p.d.e. to be integrable in more than two dimensions. So I would like to bring this question to the attention of the integrable systems community, in the hope that someone else may be able to answer it.

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